OSCILLATORY MOTION OF A STRING SUPPORTED RIGID BODY

Bruce P Minaker
Department of Mechanical, Automotive, & Materials Engineering, University of Windsor, Windsor, ON, Canada

---

ABSTRACT

The paper explores the small motion characteristics of a string supported rigid body. The equations of motion are generated using two approaches: Lagrange's equation, and constrained Newton-Euler. The geometry is studied to determine the location of the centre of curvature of the path of the centre of mass, and its effect on the natural frequency.

Keywords: equations of motion; natural frequency; centre of curvature.

TITRE FRANÇAIS DE L'ARTICLE (MAXIMUM DEUX LIGNES)

RÉSUMÉ

L'article explore les caractéristiques de petit mouvement d'un corps rigide supporté par des cordes. Les équations de mouvement sont générés à l'aide de deux approches : l'équation de Lagrange et Newton-Euler contraint. Le la géométrie est étudiée pour déterminer l'emplacement du centre de courbure de la trajectoire du centre de masse, et son effet sur la fréquence naturelle.

Mots-clés : équations de mouvement ; fréquence naturelle ; centre de courbure.

1. INTRODUCTION

The objective of this study is to explore oscillatory motion of a string supported rigid body, like the one shown in Fig. 1. The body is assumed to be restricted to planar motion, i.e., it can move in the x-y plane, and rotate about a perpendicular z-axis. It is assumed that there are two strings restricting the motion, and that both remain taut (or equivalently, that they are rigid and can support a compressive load), and that as a result, the system has a single degree of freedom. The problem is assumed to be well defined, e.g., the geometry of the strings is such that any singular configurations or limits of motion are avoided. Further, the motion is defined as measured from an equilibrium condition, i.e., when $x = y = \theta = 0$, the body is in equilibrium, and lastly, small motions are considered, such that the resulting equation of motion can be linearized.

The equations of motion are generated using two approaches: Lagrange's equation, and constrained Newton-Euler, and the results are compared. The geometry of the problem is studied to determine the location of the centre of curvature of the path of the centre of mass, and its effect on the natural frequency.

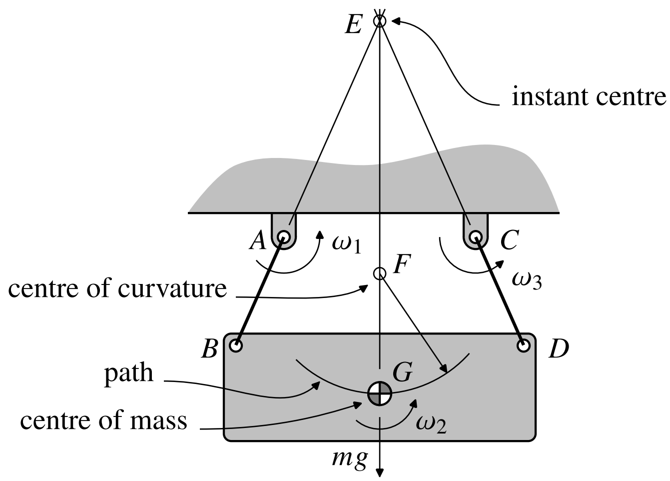


Fig. 1. A mass suspended by strings, showing the instant centre (centre of zero velocity) at point H, and the centre of curvature (point F) of the path of the centre of mass (point G).

2. LAGRANGIAN APPROACH

Lagrange's well-known equation of motion can be used to find the equation of motion of a body or a system of bodies. For a detailed discussion of Lagrange's equation, see Greenwood[1]. For a conservative system like the one considered, the external forces can be ignored, and the equation becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{1}$$

The Lagrangian (L) is the difference between the kinetic energy (T) and the potential energy (V), and q is a generalized coordinate. There is one generalized coordinate assigned for each degree of freedom.

$$L = T - V \tag{2}$$

In order to develop expressions for L and V, an analysis of the kinematics follows.

2.1. Kinematic analysis

The analysis begins with the definition of the orthogonal operator, as given in Eq. (3). If a vector $\boldsymbol{v}$ is defined as:

$$\boldsymbol{v} = \begin{Bmatrix} v_x \\ v_y \end{Bmatrix} \Rightarrow \boldsymbol{v}^{\perp} = \begin{Bmatrix} -v_y \\ v_x \end{Bmatrix} \tag{3}$$

If the vector $\boldsymbol{r}$ lies in the plane of interest, and the vector $\boldsymbol{\omega}$ is perpendicular to this plane, then the orthogonal operator allows the expression of the cross product as:

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} = \omega \boldsymbol{r}^{\perp} \tag{4}$$

where $\omega = |\boldsymbol{\omega}|$. Also, considering the dot product:

$$\boldsymbol{v} \cdot \boldsymbol{v}^{\perp} = \boldsymbol{v}' \boldsymbol{v}^{\perp} = 0 \tag{5}$$

Once the notation has been defined, a standard relative velocity equation is formed for the points B and D (as defined in Fig. 1):

$$\boldsymbol{v}_B + \boldsymbol{v}_{D/B} = \boldsymbol{v}_D \tag{6}$$

or equivalently:

$$\omega_1 \boldsymbol{r}^{\perp}_{B/A} + \omega_2 \boldsymbol{r}^{\perp}_{D/B} = \omega_3 \boldsymbol{r}^{\perp}_{D/C} \tag{7}$$

where ω_1, ω_2, and ω_3 are the angular velocities defined in Fig. 1. Gathering the angular velocities into a vector gives:

$$\begin{bmatrix} \boldsymbol{r}^{\perp}_{B/A} & \boldsymbol{r}^{\perp}_{D/B} \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \end{Bmatrix} = \omega_3 \boldsymbol{r}^{\perp}_{D/C} \tag{8}$$

Two angular velocities of can be expressed as a linear function of the third.

$$\begin{Bmatrix} \omega_1 \\ \omega_2 \end{Bmatrix} = \begin{bmatrix} \boldsymbol{r}^{\perp}_{B/A} & \boldsymbol{r}^{\perp}_{D/B} \end{bmatrix}^{-1} \omega_3 \boldsymbol{r}^{\perp}_{D/C} = \begin{Bmatrix} a \\ b \end{Bmatrix} \omega_3 \tag{9}$$

A similar expression can be applied to find the velocity of the centre of mass G.

$$\begin{aligned} \boldsymbol{v}_G &= \boldsymbol{v}_B + \boldsymbol{v}_{G/B} \\ &= \omega_1 \boldsymbol{r}^{\perp}_{B/A} + \omega_2 \boldsymbol{r}^{\perp}_{G/B} \\ &= \begin{bmatrix} \boldsymbol{r}^{\perp}_{B/A} & \boldsymbol{r}^{\perp}_{G/B} \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \end{Bmatrix} \\ &= \begin{bmatrix} \boldsymbol{r}^{\perp}_{B/A} & \boldsymbol{r}^{\perp}_{G/B} \end{bmatrix} \begin{bmatrix} \boldsymbol{r}^{\perp}_{B/A} & \boldsymbol{r}^{\perp}_{D/B} \end{bmatrix}^{-1} \omega_3 \boldsymbol{r}^{\perp}_{D/C} = \boldsymbol{r}_1 \omega_3 \end{aligned} \tag{10}$$

where:

$$\boldsymbol{r}_1 = \begin{bmatrix} \boldsymbol{r}^{\perp}_{B/A} & \boldsymbol{r}^{\perp}_{G/B} \end{bmatrix} \begin{bmatrix} \boldsymbol{r}^{\perp}_{B/A} & \boldsymbol{r}^{\perp}_{D/B} \end{bmatrix}^{-1} \boldsymbol{r}^{\perp}_{D/C} \tag{11}$$

Note that the $\boldsymbol{r}_1$ vector must be tangent to the path of point G, i.e., $\boldsymbol{r}_1 = r_1 \hat{\boldsymbol{u}}_{\mathrm{t}}$, and that:

$$\boldsymbol{r}_{G/E} = \frac{\boldsymbol{r}_1^{\perp}}{b} \tag{12}$$

where the point E is the instant center of motion of the body. In cases where the angular velocity of the body is zero (e.g., the strings are parallel), the value $b = 0$, and the distance to the instant centre is undefined. The acceleration problem is solved next:

$$\boldsymbol{a}_B + \boldsymbol{a}_{D/B} = \boldsymbol{a}_D \tag{13}$$

or equivalently:

$$\alpha_1 \boldsymbol{r}^{\perp}_{B/A} - \omega_1^2 \boldsymbol{r}_{B/A} + \alpha_2 \boldsymbol{r}^{\perp}_{D/B} - \omega_2^2 \boldsymbol{r}_{D/B} = \alpha_3 \boldsymbol{r}^{\perp}_{D/C} - \omega_3^2 \boldsymbol{r}_{D/C} \tag{14}$$

Gathering two angular accelerations gives:

$$\begin{bmatrix} \boldsymbol{r}^{\perp}_{B/A} & \boldsymbol{r}^{\perp}_{D/B} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \alpha_3 \boldsymbol{r}^{\perp}_{D/C} - \omega_3^2 \boldsymbol{r}_{D/C} + \omega_1^2 \boldsymbol{r}_{B/A} + \omega_2^2 \boldsymbol{r}_{D/B} \tag{15}$$

Solving for the angular accelerations gives:

$$\begin{aligned} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} &= \begin{bmatrix} \boldsymbol{r}^{\perp}_{B/A} & \boldsymbol{r}^{\perp}_{D/B} \end{bmatrix}^{-1} \left\{ \alpha_3 \boldsymbol{r}^{\perp}_{D/C} - \omega_3^2 \boldsymbol{r}_{D/C} + \omega_1^2 \boldsymbol{r}_{B/A} + \omega_2^2 \boldsymbol{r}_{D/B} \right\} \\ &= \begin{bmatrix} \boldsymbol{r}^{\perp}_{B/A} & \boldsymbol{r}^{\perp}_{D/B} \end{bmatrix}^{-1} \left\{ \alpha_3 \boldsymbol{r}^{\perp}_{D/C} - \omega_3^2 \boldsymbol{r}_{D/C} + \begin{bmatrix} \boldsymbol{r}_{B/A} & \boldsymbol{r}_{D/B} \end{bmatrix} \begin{Bmatrix} \omega_1^2 \\ \omega_2^2 \end{Bmatrix} \right\} \end{aligned} \tag{16}$$

The acceleration of point G can be written:

$$\begin{aligned} \boldsymbol{a}_G &= \boldsymbol{a}_B + \boldsymbol{a}_{G/B} \\ &= -\omega_1^2 \boldsymbol{r}_{B/A} + \alpha_1 \boldsymbol{r}^{\perp}_{B/A} - \omega_2^2 \boldsymbol{r}_{G/B} + \alpha_2 \boldsymbol{r}^{\perp}_{G/B} \\ &= \begin{bmatrix} \boldsymbol{r}^{\perp}_{B/A} & \boldsymbol{r}^{\perp}_{G/B} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} - \begin{bmatrix} \boldsymbol{r}_{B/A} & \boldsymbol{r}_{G/B} \end{bmatrix} \begin{Bmatrix} \omega_1^2 \\ \omega_2^2 \end{Bmatrix} \\ &= \boldsymbol{r}_1 \alpha_3 + \boldsymbol{r}_2 \omega_3^2 \end{aligned} \tag{17}$$

where:

$$\boldsymbol{r}_2 = \begin{bmatrix} \boldsymbol{r}^{\perp}_{B/A} & \boldsymbol{r}^{\perp}_{G/B} \end{bmatrix} \begin{bmatrix} \boldsymbol{r}^{\perp}_{B/A} & \boldsymbol{r}^{\perp}_{D/B} \end{bmatrix}^{-1} \left[\begin{bmatrix} \boldsymbol{r}_{B/A} & \boldsymbol{r}_{D/B} \end{bmatrix} \begin{Bmatrix} a^2 \\ b^2 \end{Bmatrix} - \boldsymbol{r}_{D/C} \right] - \begin{bmatrix} \boldsymbol{r}_{B/A} & \boldsymbol{r}_{G/B} \end{bmatrix} \begin{Bmatrix} a^2 \\ b^2 \end{Bmatrix} \tag{18}$$

Next, an expression for the normal component of acceleration of point G can be written as a component of the total acceleration.

$$\boldsymbol{a}_{\mathrm{n}} = (\boldsymbol{a}_G \cdot \hat{\boldsymbol{u}}_{\mathrm{n}}) \hat{\boldsymbol{u}}_{\mathrm{n}} \tag{19}$$

The same acceleration can also be written in terms of the velocity of point G and the location of point F (the centre of curvature of the path of point G). One might expect that point F and point E share the same location, but a simple contra-example is the case of parallel equal length strings, where point E is infinitely far away, but the distance from G to F must be just the length of the strings.

$$\boldsymbol{a}_{\mathrm{n}} = \frac{|\boldsymbol{v}_G|^2}{|\boldsymbol{r}_{F/G}|} \hat{\boldsymbol{u}}_{\mathrm{n}} \tag{20}$$

Equating the expressions in Eq. (19) and (20) and rearranging gives an expression for the location of point F.

$$|\boldsymbol{r}_{F/G}| = \frac{|\boldsymbol{v}_G|^2}{\boldsymbol{a}_G \cdot \hat{\boldsymbol{u}}_\mathrm{n}} \tag{21}$$

Substituting the expressions for the velocity and acceleration of point G gives:

$$\boldsymbol{r}_{F/G} = |\boldsymbol{r}_{F/G}|\hat{\boldsymbol{u}}_\mathrm{n} = \frac{r_1^2\omega_3^2}{\left(r_1\alpha_3\hat{\boldsymbol{u}}_\mathrm{t} + \boldsymbol{r}_2\omega_3^2\right)\cdot\hat{\boldsymbol{u}}_\mathrm{n}}\hat{\boldsymbol{u}}_\mathrm{n} = \frac{r_1^2\omega_3^2}{\boldsymbol{r}_2\omega_3^2\cdot\hat{\boldsymbol{u}}_\mathrm{n}}\hat{\boldsymbol{u}}_\mathrm{n} = \frac{r_1^2}{\boldsymbol{r}_2\cdot\hat{\boldsymbol{u}}_\mathrm{n}}\hat{\boldsymbol{u}}_\mathrm{n} \tag{22}$$

Note that, as expected, the curvature of the path of G is not a function of the velocity or acceleration of the body, so the angular velocity and angular acceleration terms cancel out of the expression. The result also implies that the vectors from the point G to the centre of curvature F and to the instant centre H must be co-linear. Because the location of the centre of curvature is independent of the angular velocity and acceleration, one can simplify the calculation in practice by assuming $\omega_3 = 1$ and $\alpha_3 = 0$, and noting that in this case, $\boldsymbol{v}_G = \boldsymbol{r}_1$ and $\boldsymbol{a}_G = \boldsymbol{r}_2$.

2.2. Energy expressions and the Lagrangian

Once the locations of points E and F have been found, expressions for the kinetic and potential energy can be formed. To simplify the notation, let $|\boldsymbol{r}_{G/F}| = r_{G/F}$ and $|\boldsymbol{r}_{G/E}| = r_{G/E}$. The potential energy is found by assuming that the body moves on an arc as defined by the centre of curvature F, with angular displacement of θ_4.

$$V = mgr_{G/F}(1 - \cos\theta_4) \tag{23}$$

Conversely, the kinetic energy depends on the relative contribution of the linear and angular velocities, which is set by the location of the instant centre E.

$$T = \frac{1}{2}I_E\omega_2^2 = \frac{1}{2}\left(I_G + mr_{G/E}^2\right)\dot{\theta}_2^2 \tag{24}$$

The generalized coordinate q will be chosen as θ_2. Note that when evaluating the derivatives for Lagrange's equations, one is faced with the terms for the rate of change of $r_{G/E}$ and $r_{G/F}$ with respect to θ_2. The static equilibrium condition requires that both points E and F must be directly vertically above point G, and further that the tangent to the path of all three points is exactly horizontal. The distance between the points is not constant, but is at a minimum when the body is in equilibrium, which implies that the rate of change of must be instantaneously zero. This allows a significant simplification to the equation of motion. Computing the first term of Lagrange's equation:

$$\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{\theta}_2} = \frac{\partial T}{\partial \dot{\theta}_2} - \cancelto{0}{\frac{\partial V}{\partial \dot{\theta}_2}} = \left(I_G + mr_{G/E}^2\right)\dot{\theta}_2 \tag{25}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{q}}\right) = \left(I_G + mr_{G/E}^2\right)\ddot{\theta}_2 \tag{26}$$

To compute the result for potential energy, a relation for θ_4 is needed. Equating the displacement of G, for small motions, gives:

$$r_{G/E}\partial\theta_2 = r_{G/F}\partial\theta_4 \tag{27}$$

or:

$$\frac{r_{G/E}}{r_{G/F}} = \frac{\partial\theta_4}{\partial\theta_2} \tag{28}$$

So, for Lagrange:

$$\begin{aligned}\frac{\partial L}{\partial q} = \frac{\partial L}{\partial\theta_2} = \cancelto{0}{\frac{\partial T}{\partial\theta_2}} - \frac{\partial V}{\partial\theta_2} &= -mgr_{G/F}\sin\theta_4\frac{\partial\theta_4}{\partial\theta_2} \\ &= -mgr_{G/F}\sin\theta_4\frac{r_{G/E}}{r_{G/F}} \\ &\approx -mgr_{G/E}\theta_4 \\ &\approx -mg\frac{r^2_{G/E}}{r_{G/F}}\theta_2\end{aligned} \tag{29}$$

The resulting equation of motion is then:

$$\left(I_G + mr^2_{G/E}\right)\ddot{\theta}_2 + mg\frac{r^2_{G/E}}{r_{G/F}}\theta_2 = 0 \tag{30}$$

Note the relationship to the famous Euler-Savary equation, where the stiffness term as shown is directly the distance from the centre of mass to a point on the inflection circle. For more information on the Euler-Savary equation, see Hartenberg and Denavit[2]. Dividing through by the square of the distance to the instant centre:

$$\left(\frac{I_G}{r^2_{G/E}} + m\right)\ddot{\theta}_2 + \frac{mg}{r_{G/F}}\theta_2 = 0 \Rightarrow \omega_n = \sqrt{\frac{mg}{r_{G/F}\left(\frac{I_G}{r^2_{G/E}} + m\right)}} \tag{31}$$

An examination of the result shows some interesting points. If the instant centre is infinitely distant, as in the case of parallel strings, the motion of the body is purely translational, and the effect of the moment of inertia should disappear, as it does in Eq. (31). Also, the radius of curvature appears in the denominator of the stiffness term. If one considers a case with equal length parallel strings, the period of the oscillation would be inversely proportional to the square root of the length of the string, as it is in a simple pendulum.

3. CONSTRAINED NEWTON-EULER EQUATION APPROACH

The equations of motion of the string supported body can be generated using the Newton-Euler approach for comparison. The method begins with the definition of coordinates. The position vector $\boldsymbol{p}$ is defined as the location of the mass centre G, and the orientation θ around an axis perpendicular to the plane of motion.

$$\boldsymbol{p} = \begin{Bmatrix} x_G \\ y_G \\ \theta \end{Bmatrix} \tag{32}$$

The Newton-Euler equations are written together as:

$$\mathbf{M}\ddot{\boldsymbol{p}} = \sum \boldsymbol{f}(\boldsymbol{p}, \boldsymbol{w}, t) \tag{33}$$

The mass matrix $\mathbf{M}$ is formed by filling its diagonal with the mass and inertia values.

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_z \end{bmatrix} \tag{34}$$

The forces are sorted into a number of different categories: inertial, elastic, constraint, and applied. The inertial forces are the centripetal forces and gyroscopic moments that appear in the Newton-Euler equations; expressions for these terms are well known. They do not appear in this scenario and are ignored. The elastic forces are generated by 'flexible' joints, e.g., a spring or bushing; they also do not appear in this scenario and are ignored. The constraint forces $\boldsymbol{f}_\mathrm{c}$ are those applied by 'rigid' connections, i.e., those that reduce the degrees of freedom of the system, e.g., a hinge or a string. The applied forces $\boldsymbol{f}_\mathrm{a}$ are anything external to the system. In this case, the weight is considered an applied force. The equation of motion can be rewritten as:

$$\mathbf{M}\ddot{\boldsymbol{p}} = \sum \boldsymbol{f}_\mathrm{c} + \sum \boldsymbol{f}_\mathrm{a} \tag{35}$$

To linearize the Newton-Euler equations, a variation is taken.

$$\mathbf{M}\delta\ddot{\boldsymbol{p}} = \sum \delta\boldsymbol{f}_\mathrm{c} + \sum \delta\boldsymbol{f}_\mathrm{a} \tag{36}$$

3.1. Constraints

Expressions for the variation of the forces are needed. The constraint forces are considered first. The rigid connectors are modelled using algebraic constraint equations; expressions for the constraint forces will be developed from these constraint equations. The constraint equation is:

$$\boldsymbol{\phi}(\boldsymbol{p}) = \mathbf{0} \tag{37}$$

Each row of the $\boldsymbol{\phi}$ vector represents a single constraint, so most mechanical connectors will require more than one row. The string constraint removes only a single degree of freedom, so each string requires one row. The constraint equations can be linearized by considering the motions to be small. Taking a variation gives:

$$\delta\boldsymbol{\phi} = \frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{p}}\delta\boldsymbol{p} = \mathbf{J}\delta\boldsymbol{p} = \mathbf{0} \tag{38}$$

The partial derivative of the constraint equations with respect to the coordinates is the constraint Jacobian matrix $\mathbf{J}$. The deflection of the rigid elements is expressed as a linear function of the motion coordinates, and must be zero. In this scenario, only holonomic constraints are considered. The constraint Jacobian matrix for a single string is:

$$\mathbf{J} = \begin{bmatrix} u_x & u_y & r_x u_y - r_y u_x \end{bmatrix} \tag{39}$$

where the string direction is defined by a unit vector $\hat{\boldsymbol{u}}$, and the location of the point where it is attached to the body is given by $\boldsymbol{r}$. The terms u_x, u_y represent the components of the unit vector, and similar for r_x, r_y. The constraint equation enforces the requirement that the motion of the body at the point where the string attaches to it must be perpendicular to the direction of the string. In this case, because there are two strings, each one will contribute one row to the constraint matrix. The constraint forces can be expressed using the Jacobian matrix:

$$\boldsymbol{f}_\mathrm{c} = -\mathbf{J}'\boldsymbol{\lambda} \tag{40}$$

where $\boldsymbol{\lambda}$ is known as a *Lagrange multiplier*. This equation stems from the fact that a motion in a particular direction can only be resisted by a force in the same direction. For more information on Lagrange multipliers, see Greenwood[1]. The Lagrange multiplier scales a vector in each direction by some unknown amount. In the example of the string, each unknown Lagrange multiplier would be the force in a string. The Jacobian matrix would serve to transform those string forces into the components in each direction, and compute the resulting moment around the mass centre as well. The unknown values of the vector $\boldsymbol{\lambda}$ are evaluated at equilibrium, so the tangent stiffness matrix requires that the preloads be determined prior to generation of the equations of motion, i.e., using static equilibrium. If there is no preload in the mechanism at equilibrium, the tangent stiffness term evaluates to zero. The constraint forces are linearized by taking a variation.

$$
\begin{aligned}
\delta \boldsymbol{f}_{\mathrm{c}} &= -\delta \mathbf{J}' \boldsymbol{\lambda} - \mathbf{J}' \delta \boldsymbol{\lambda} \\
&= -\frac{\partial \mathbf{J}'}{\partial \boldsymbol{p}} \delta \boldsymbol{p} \boldsymbol{\lambda} - \mathbf{J}' \delta \boldsymbol{\lambda}
\end{aligned}
\tag{41}
$$

The constraints generate a tangent stiffness matrix in the event that there are any preload forces carried when the system is in equilibrium. It is defined as:

$$
\mathbf{K}_{\mathrm{c}} = \frac{\partial \mathbf{J}'}{\partial \boldsymbol{p}} \boldsymbol{\lambda}
\tag{42}
$$

so the constraint forces can be expressed as:

$$
\delta \boldsymbol{f}_{\mathrm{c}} = -\mathbf{K}_{\mathrm{c}} \delta \boldsymbol{p} - \mathbf{J}' \delta \boldsymbol{\lambda}
\tag{43}
$$

The tangent stiffness term can be added to any other stiffness terms to contribute to a total stiffness matrix. Any changes in the applied forces are treated as functions of the position vector, or its derivative. In this case, there is no damping, so only the stiffness terms are needed.

$$
\delta \boldsymbol{f}_{\mathrm{a}} = \frac{\partial \boldsymbol{f}_{\mathrm{a}}}{\partial \boldsymbol{p}} \delta \boldsymbol{p} = -\mathbf{K}_{\mathrm{a}} \delta \boldsymbol{p}
\tag{44}
$$

The resulting equation of motion is:

$$
\mathbf{M} \delta \ddot{\boldsymbol{p}} = -(\mathbf{K}_{\mathrm{c}} + \mathbf{K}_{\mathrm{a}}) \delta \boldsymbol{p} - \mathbf{J}' \delta \boldsymbol{\lambda}
\tag{45}
$$

In order to reduce the equations to a minimal set of coordinates, an *orthogonal complement* of the Jacobian is used. The complement is a matrix $\mathbf{T}$ defined such that its columns are perpendicular to the rows of the Jacobian. More precisely, it is an orthonormal basis for the null space of the Jacobian. As a result of the definition, the product of the complement and the Jacobian is zero.

$$
\mathbf{J}\mathbf{T} = \mathbf{0}
\tag{46}
$$

The complement matrix is not unique. For example, consider the case of a single vector in three dimensions. The null space is the plane that is normal to that vector. If two perpendicular vectors in the normal plane are sought, an infinite number of choices exist, distinguished by a rotation around the normal vector. There are several methods for calculating a suitable choice, e.g., the null space can be found using singular value decomposition. For more discussion on coordinate reduction, see Amirouche[3] or Shabana[4].

A new coordinate vector $\boldsymbol{x}$ is defined using the complement. It should be noted that this vector is now in nonphysical coordinates, i.e., the values may not directly correspond to the physical motions, although in some cases, a subset of the physical coordinates may be suitable. In any case, the physical coordinates can be recovered using the complement matrix.

$$\mathbf{T}\boldsymbol{x} = \delta \boldsymbol{p} \tag{47}$$

As a result of this definition, any possible selection of the coordinate vector must satisfy the constraint equations.

$$\mathbf{J}\mathbf{T}\boldsymbol{x} = \mathbf{J}\delta \boldsymbol{p} = \mathbf{0} \tag{48}$$

The premultiplication by the orthogonal complement reduces the number of equations to match the number of coordinates, and eliminates the remaining constraint force terms from the equations.

$$\mathbf{T}'\mathbf{M}\delta \ddot{\boldsymbol{p}} + \mathbf{T}'\mathbf{K}\delta \boldsymbol{p} = -\mathbf{T}'\mathbf{J}'\delta \boldsymbol{\lambda} = \mathbf{0} \tag{49}$$

Finally, the new coordinates can be substituted into the equations of motion.

$$\mathbf{T}'\mathbf{M}\mathbf{T}\ddot{\boldsymbol{x}} + \mathbf{T}'\mathbf{K}\mathbf{T}\boldsymbol{x} = \mathbf{0} \tag{50}$$

3.2. Form of the stiffness matrix

Consider a string that connects the body to the ground, and maintains a fixed distance between its endpoints. Now the stiffness matrix can be written by expanding expressions for the rate of change of force and moment with respect to translation and rotation of the body.

$$\mathbf{K}_{\mathrm{c}} = -\frac{\partial \boldsymbol{f}_{\mathrm{c}}}{\partial \boldsymbol{p}} = -\begin{bmatrix} \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} & \frac{\partial \boldsymbol{f}}{\partial \theta} \\ \frac{\partial m}{\partial \boldsymbol{x}} & \frac{\partial m}{\partial \theta} \end{bmatrix} \tag{51}$$

The location of the moving end of the string is defined as $\boldsymbol{x}_{\mathrm{B}}$, and the fixed end as $\boldsymbol{x}_{\mathrm{A}}$. The location $\boldsymbol{x}_{\mathrm{B}}$ can be found from the location $\boldsymbol{x}$ of the body, and the distance $\boldsymbol{r}$ of the end point B from the centre of mass. Note that $\boldsymbol{r}$ is defined as fixed in the rotating reference frame, and the rotation is assumed to be small.

$$\boldsymbol{x}_{\mathrm{B}} = \boldsymbol{x} + \boldsymbol{r} + \theta \boldsymbol{r}^{\perp} \tag{52}$$

The rate of change of location of the end of the string with respect to both the translation and rotation of the body can be found by differentiation. As expected, the motion of the end of the string is the same as that of the mass centre when there is no rotation, and that the contribution of rotation depends on only $\boldsymbol{r}$.

$$\frac{\partial \boldsymbol{x}_{\mathrm{B}}}{\partial \boldsymbol{x}} = \mathbf{I} \tag{53}$$

$$\frac{\partial \boldsymbol{x}_{\mathrm{B}}}{\partial \theta} = \boldsymbol{r}^{\perp} \tag{54}$$

The length of the string is determined from the locations of its end-points.

$$l = |\boldsymbol{x}_{\mathrm{A}} - \boldsymbol{x}_{\mathrm{B}}| \tag{55}$$

A unit vector acting along the string defines the direction. The unit vector is expressed in global coordinates, as are the positions of the end-points of the spring.

$$\hat{\boldsymbol{u}} = \frac{\boldsymbol{x}_{\mathrm{A}} - \boldsymbol{x}_{\mathrm{B}}}{l} \tag{56}$$

Note that the force must be expressed in the body fixed coordinate system, as this is the system in which the equations of motion are written. The inverse rotation matrix is used to convert the force vector from the global frame to the body fixed frame. The magnitude of the force, being a constant scalar, can be factored out in front, and the rotation matrix can be expanded.

$$\boldsymbol{f} = f\hat{\boldsymbol{u}} - f\theta\boldsymbol{u}^{\perp} \tag{57}$$

The first term in the stiffness matrix can now be expanded using the product rule.

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = f\frac{\partial \hat{\boldsymbol{u}}}{\partial \boldsymbol{x}} - f\frac{\partial \theta}{\partial \boldsymbol{x}}\boldsymbol{u}^{\perp} - f\theta\frac{\partial \boldsymbol{u}^{\perp}}{\partial \boldsymbol{x}} \tag{58}$$

When considering the resulting expression, the second and third terms can be neglected. The second term will go to zero when the expression is evaluated at $\theta = 0$, and the third will go to zero, as θ is independent of $\boldsymbol{x}$. Expanding the remaining derivative terms allows an algebraic expression for the first term in the stiffness matrix to be found.

$$\begin{aligned}\frac{\partial \hat{\boldsymbol{u}}}{\partial \boldsymbol{x}} &= \frac{\partial \hat{\boldsymbol{u}}}{\partial \boldsymbol{x}_{\mathrm{B}}}\frac{\partial \boldsymbol{x}_{\mathrm{B}}}{\partial \boldsymbol{x}} = \frac{\partial \hat{\boldsymbol{u}}}{\partial \boldsymbol{x}_{\mathrm{B}}} \\ &= -\frac{1}{l}\mathbf{I} + (\boldsymbol{x}_{\mathrm{A}} - \boldsymbol{x}_{\mathrm{B}})\frac{\hat{\boldsymbol{u}}'}{l^2} = -\frac{1}{l}[\mathbf{I} - \hat{\boldsymbol{u}}\hat{\boldsymbol{u}}'] = -\frac{1}{l}\boldsymbol{u}^{\perp}\boldsymbol{u}^{\perp\prime}\end{aligned} \tag{59}$$

Substitution gives:

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = -\boldsymbol{u}^{\perp}\frac{f}{l}\boldsymbol{u}^{\perp\prime} \tag{60}$$

The second term in the stiffness matrix can now be expanded using the product rule.

$$\boldsymbol{f} = f\hat{\boldsymbol{u}} - f\theta\boldsymbol{u}^{\perp} \tag{61}$$

$$\frac{\partial \boldsymbol{f}}{\partial \theta} = f\frac{\partial \hat{\boldsymbol{u}}}{\partial \theta} - f\boldsymbol{u}^{\perp} - f\frac{\partial \boldsymbol{u}^{\perp}}{\partial \theta}\theta \tag{62}$$

Again, the third term in the expression can be neglected as it will go to zero when evaluated at $\theta = 0$. The rate of change of the force relative to rotation can be found in terms of the force relative to translation.

$$\frac{\partial \hat{\boldsymbol{u}}}{\partial \theta} = \frac{\partial \hat{\boldsymbol{u}}}{\partial \boldsymbol{x}_{\mathrm{B}}}\frac{\partial \boldsymbol{x}_{\mathrm{B}}}{\partial \theta} = \frac{\partial \hat{\boldsymbol{u}}}{\partial \boldsymbol{x}}\boldsymbol{r}^{\perp} \tag{63}$$

Substituting gives:

$$\frac{\partial \boldsymbol{f}}{\partial \theta} = -\boldsymbol{u}^{\perp}\frac{f}{l}\boldsymbol{u}^{\perp\prime}\boldsymbol{r}^{\perp} - \boldsymbol{f}^{\perp} \tag{64}$$

Finally, the change of moment terms can be found.

$$\boldsymbol{m} = \boldsymbol{r}^{\perp\prime}\boldsymbol{f} \tag{65}$$

Recognizing that $\boldsymbol{r}$ is constant and substituting the expressions for force gives:

$$\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{x}} = \boldsymbol{r}^{\perp\prime}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = -\boldsymbol{r}^{\perp\prime}\boldsymbol{u}^{\perp}\frac{f}{l}\boldsymbol{u}^{\perp\prime} \tag{66}$$

$$\frac{\partial \boldsymbol{m}}{\partial \theta} = \boldsymbol{r}^{\perp\prime}\frac{\partial \boldsymbol{f}}{\partial \theta} = \boldsymbol{r}^{\perp\prime}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\boldsymbol{r}^{\perp\prime} - \boldsymbol{r}^{\perp\prime}\boldsymbol{f}^{\perp} \tag{67}$$

The stiffness matrix is now complete.

$$\mathbf{K}_c = \begin{bmatrix} \mathbf{0} & \boldsymbol{f}^\perp \\ \mathbf{0} & \boldsymbol{r}^{\perp\prime}\boldsymbol{f}^\perp \end{bmatrix} + \begin{bmatrix} \boldsymbol{u}^\perp \frac{f}{l} \boldsymbol{u}^{\perp\prime} & \boldsymbol{u}^\perp \frac{f}{l} \boldsymbol{u}^{\perp\prime} \boldsymbol{r}^\perp \\ sym & \boldsymbol{r}^{\perp\prime} \boldsymbol{u}^\perp \frac{f}{l} \boldsymbol{u}^{\perp\prime} \boldsymbol{r}^\perp \end{bmatrix} \tag{68}$$

The stiffness matrix can be expanded as the sum of two components, one of which can be factored as below.

$$\begin{aligned} \mathbf{K}_c &= \begin{bmatrix} \mathbf{0} & \boldsymbol{f}^\perp \\ \mathbf{0} & \boldsymbol{r}'\boldsymbol{f} \end{bmatrix} + \frac{f}{l} \begin{bmatrix} \boldsymbol{u}^\perp \\ \boldsymbol{r}'\boldsymbol{u} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^\perp \\ \boldsymbol{r}'\boldsymbol{u} \end{bmatrix}' \\ &= f \begin{bmatrix} 0 & 0 & -u_y \\ 0 & 0 & u_x \\ 0 & 0 & r_x u_x + r_y u_y \end{bmatrix} + \frac{f}{l} \begin{bmatrix} u_y^2 & -u_x u_y & -u_y(r_x u_x + r_y u_y) \\ -u_x u_y & u_x^2 & u_x(r_x u_x + r_y u_y) \\ -u_y(r_x u_x + r_y u_y) & u_x(r_x u_x + r_y u_y) & (r_x u_x + r_y u_y)^2 \end{bmatrix} \end{aligned} \tag{69}$$

A careful examination of the two terms shows that the first one is due to the change in direction of the string force relative to the body fixed frame as the body reorients, while the second is due to the change in direction of the force as the string reorients. Not included in Eq. (69) is the effect of the weight force, which must be computed separately. The weight force reorients relative to the body fixed rotating frame. The effect is equivalent to a string of infinite length with a zero length radius vector.

4. EXAMPLE

Variations on a sample geometry are used to illustrate the result from each of the two methods. In all cases, the mass $m = 24$ kg, the moment of inertia $I_G = 2.5$ kg m^2, and the gravitational constant is taken as $g = 9.81$ m/s^2. The geometry of the four variations is shown in Fig. 2 and given in Table 1. In all four cases, the results of both methods agreed to ten significant figures or better. The four variations are:

a) Example 1: strings are parallel, equal length, and equally loaded. In this case, the instant centre is infinitely far away, so there is no rotation of the body. The radius of curvature of the path of the mass centre is simply the length of both strings.

b) Example 2: strings are parallel, unequal length, and equally loaded. Again, the instant centre is infinitely far away, so there is no rotation of the body. The radius of curvature of the path of the mass centre is more challenging to find.

c) Example 3: strings are not parallel, equal length, equally loaded. The instant centre is a finite distance from the mass centre, so the moment of inertia now has an influence on the result. The radius of curvature of the path of the mass centre again requires calculation.

d) Example 4: Strings are not parallel, unequal length, and unequally loaded. The instant centre is the same distance from the mass centre as in case *c*), but the radius of curvature has changed.

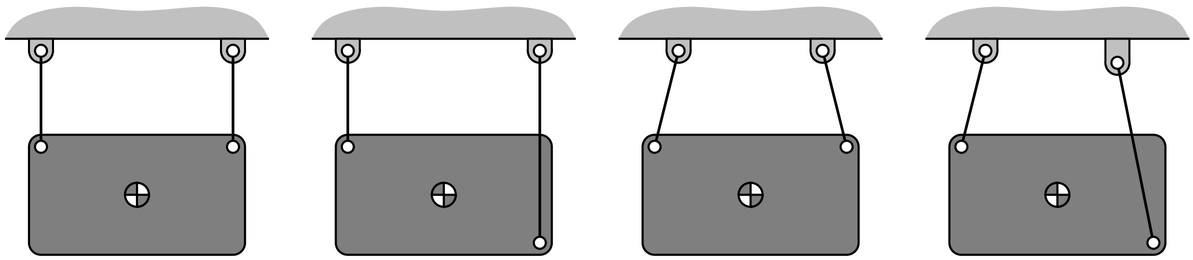
The matrix calculation for the first configuration is shown. The mass matrix is:

$$\mathbf{M} = \begin{bmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 2.5 \end{bmatrix} \tag{70}$$

The stiffness matrix, including the contributions from each string and from the weight force, is:

$$\mathbf{K} = \begin{bmatrix} 470.88 & 0.0 & -117.72 \\ 0.0 & 0.0 & 0.0 \\ -117.72 & 0.0 & 88.29 \end{bmatrix} \tag{71}$$

The Jacobian constraint matrix and its complement are:



(a) Example 1 (b) Example 2 (c) Example 3 (d) Example 4

Fig. 2. The equations of motion of the system were generated using both methods in four configurations.

Location	Example 1	Example 2	Example 3	Example 4
$\boldsymbol{r}_A$	(-0.5, 0.75) m	(-0.5, 0.75) m	(-0.375, 0.75) m	(-0.375, 0.75) m
$\boldsymbol{r}_B$	(-0.5, 0.25) m	(-0.5, 0.25) m	(-0.5, 0.25) m	(-0.5, 0.25) m
$\boldsymbol{r}_C$	(0.5, 0.75) m	(0.5, 0.75) m	(0.375, 0.75) m	(0.3125, 0.6875) m
$\boldsymbol{r}_D$	(0.5, 0.25) m	(0.5, -0.25) m	(0.5, 0.25) m	(0.5, -0.25) m
$\boldsymbol{r}_{G/E}$	∞	∞	2.25 m	2.25 m
$\boldsymbol{r}_{G/F}$	0.5 m	0.667 m	0.587 m	0.676 m
ω_n	0.705 Hz	0.611 Hz	0.644 Hz	0.600 Hz

Table 1. Example system properties. Results computed with $m = 24$ kg, $I_G = 2.5$ kg m^2, and $g = 9.81$ m/s^2.

$$\mathbf{J} = \begin{bmatrix} 0.0 & 1.0 & -0.5 \\ 0.0 & 1.0 & 0.5 \end{bmatrix} \Rightarrow \mathbf{T} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{72}$$

The resulting equation of motion is:

$$24\ddot{x} + 470.88x = 0 \Rightarrow \omega_n = \sqrt{\frac{470.88}{24}} = 4.429 \text{ rad/s} = 0.705 \text{ Hz} \tag{73}$$

5. CONCLUSIONS

The equations of motion to predict the small motion oscillations of a string suspended body have been generated using two methods. The methods give identical results. The Lagrangian approach identifies that the centre of curvature of the path of the centre of mass is an important factor in determining the behaviour. The stiffness term of the equation of motion contains the distance from the centre of mass to centre of curvature in the denominator, implying the geometries with higher curvature will have higher natural frequencies. It is notable the two different geometries can have the same instant centre, but very different curvatures, so the location of the instant centre alone is insufficient to determine the resulting natural frequency.

REFERENCES

1. Greenwood, D.T. *Principles of dynamics*. Prentice-Hall Englewood Cliffs, NJ, 1988.
2. Hartenberg, R.S. and Denavit, J. *Kinematic synthesis of linkages*. McGraw-Hill, 1964.
3. Amirouche, F.M. *Fundamentals of multibody dynamics: theory and applications*. Springer, 2006.
4. Shabana, A.A. *Computational dynamics*. John Wiley & Sons, 2009.