# MODELLING AND ANALYSIS OF A HYDRODYNAMICALLY ACTUATED COUPLED PDE-ODE SYSTEM

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## ABSTRACT

Defence Research and Development Canada (DRDC), in collaboration with the University of New Brunswick (UNB) and other institutions, have proposed a program to develop an autonomous docking device capable of docking and retrieving a UUV (Underwater Unmanned Vehicles) from a submerged submarine. Given the relatively slow dynamics of most off-the-shelf UUVs, the complexity of the capture task will require the active dock to be dexterous and autonomous to accomplish the task. In this paper, mathematical modelling is proposed in order to use it for analysis and control synthesis. A coupled PDE (Partial Differential Equation) and an ODE (Ordinary Differential Equation) is proposed in order to describe the behaviour of a hydrodynamically actuated manipulator. In order to describe the fluid-solid dynamics, the linear advection equation coupled with a linear ODE is considered to describe an underwater manipulator. The mathematical model considers a flow interacting with a mass-spring-damper, and it is compared with an analytical solution of the PDE and the classic modelling for underwater manipulators such as added mass and drag friction. This qualitative comparison is shown through numerical simulations using an underwater MSD (Mass-Spring-Damper) as a test bed.

**Keywords:** Modelling; hydrodynamic system; nonlinear systems; numerical methods.

# MODÉLISATION ET ANALYSE D'UN SYSTÈME ACTIONNÉ DE MANIÈRE HYDRODYNAMIQUE COUPLÉ DE TYPE EDP-EDO

## RÉSUMÉ

Recherche et développement pour la défense Canada (RDDC), en collaboration avec l'Université du Nouveau-Brunswick (UNB) et d'autres institutions, ont proposé un programme visant à développer un dispositif autonome capable de réaliser l'opération d'amarrage d'un robot sous-marin autonome (UUV ou AUV) à partir d'un sous-marin submergé. La complexité du dispositif pour la capture des UUV se situera sur un quai actif, ce qui permettra d'utiliser une grande variété d'UUV disponibles et à venir. Dans cet article, une modélisation mathématique est proposée afin de l'utiliser pour l'analyse et la synthèse du logiciel de contrôle. Une EDP (équation différentielle partielle) couplée avec une EDO (équation différentielle ordinaire) est proposée afin de décrire le comportement d'un manipulateur actionné de manière hydrodynamique. Afin de décrire la dynamique entre fluides et solides, l'équation d'advection linéaire couplée à une EDO linéaire est considérée comme décrivant un manipulateur sous-marin. Le modèle mathématique considère un écoulement en interaction avec un système solide de masse-ressort-amortisseur. Il est comparé à une solution analytique de l'EDP et à la modélisation classique des manipulateurs sous-marins tels que la masse ajoutée et la traînée causée par le fluide. Cette comparaison qualitative est illustrée par des simulations numériques utilisant un système de masse-ressort-amortisseur sous-marin en guise de banc de test.

**Mots-clés :** Modélisation ; système hydrodynamique ; système non-linéaire ; méthode numérique.

## 1. INTRODUCTION

Underwater manipulators are very popular nowadays because they allow more flexibility and a wider range of applications for UUVs (Underwater Unmanned Vehicles). Moreover, once a UUV is deployed and its task is completed, a mechatronic manipulator can be used to recover it, in other words: a mobile dock to recover the remote-controlled robot [1]. In order to control an underwater manipulator, additional dynamics must be considered because the density of the water is comparable with the density of the manipulator. For high-performance manipulators, hydrodynamic forces are significant, and their effect can induce unwanted motions affecting the precision of the controlled manipulator. In the literature, hydrodynamic concepts such as added mass, drag, lift, among others have been studied for decades for analysis and control design purposes (see [2, 3]). It is well known that most floating/underwater structures can be modelled, for small motions and linear behaviour by a system equation with the basic form similar to a typical Mass-Spring-Damper (MSD) [4–6].

The lumped approach to model hydrodynamic effects on an underwater manipulator has limited accuracy and the control design would not be precise or robust against unmodelled uncertainties. These models are simple and suitable for control design and analysis purposes. However, because of the unmodelled uncertainties, they may cause problems such as instability, a high order of the controller, and difficulties in implementation (*e.g.*, [7–9]). A solution could be in the study of the dynamics in the Partial Differencial Equations (PDE). Increasing the complexity of the system, modelling Fluid-Structure Interactions (FSI) or Computational Fluid Dynamics (CFD) models would yield more accurate results. For example, for flexible manipulators (not necessarily underwater), some researches have studied control based on a PDE-based dynamic model derived without any truncation and thus could represent the spatiotemporally-varying states accurately (see [10]).

In scientific literature, considerable attention was paid to infinite dimensional systems, represented by PDE models (see for instance [11, 12]). In industrial processes, a large variety of physical systems are governed by PDEs such as hydraulic networks, gas devices, fluid dynamics, among others (*e.g.*, see [13]) while others such as pendulum or a mass-spring-damper system are governed by Ordinary Differential Equations (ODE) (*e.g.*, see [14, 15]). Moreover, in some cases, there are systems that can be described by coupled PDE-ODE dynamics, for example, the control design of a drilling process or fluid transport in a tubular membrane. In this process, the PDE is a second order linear equation while the ODE is a nonlinear fourth order equation (see [16]). However, there are still important issues to solve, for example, for control design purposes, the PDE-ODE model is sometimes very complex and the stability analysis requires complex mathematical tools (open or closed loop). Another example could be seen in computational tasks where CFD does not allow to use real-time applications due to the heavy computational calculations. In [16], the control design requires more computational effort since the PDE must be solved in order to calculate the control input.

In the field of control theory, robustness is an important issue since there are always uncertainties in the system dynamics (from no modelled dynamics to parametric uncertainties). There are several methodologies dealing with robustness such as sliding mode control, $H_\infty$, among others. These approaches consider an uncertain nonlinear system where the uncertainty can be bounded by a constant or a function. Moreover, many of them work with underactuated mechanical systems, which are also known as non-collocated dynamics. Some of these approaches consider a coupled PDE and a nonlinear ODE as a representation for a nonlinear dynamic system with a fixed boundary (see [16–18]). In other cases, a coupled PDE and a linear ODE with a dynamic boundary have been under study using these approaches (see for instance [19, 20]). However, due to the complexity of the problem, a coupled PDE with a nonlinear ODE, with a moving boundary, has not been considered to propose a robust control design for underwater underactuated manipulators (see for instance [21]).

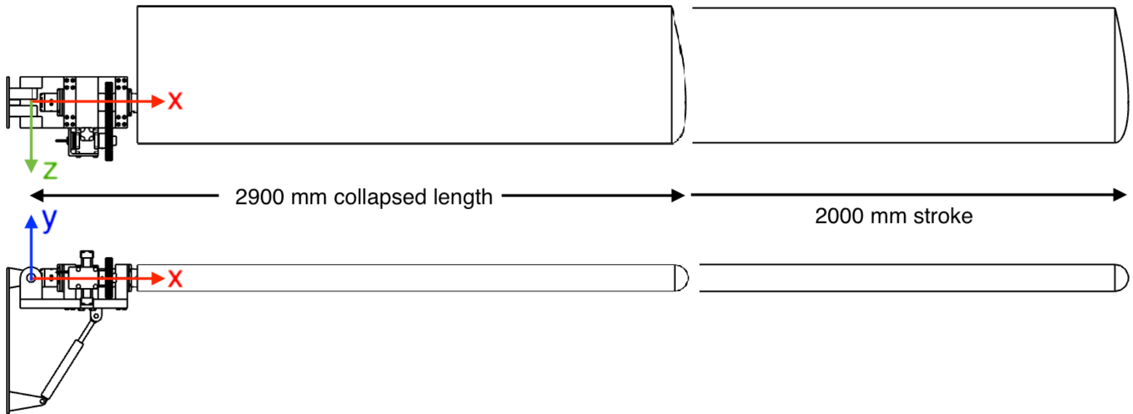

Fig. 1. The three-link mechanical system (range of motion of prismatic joint).

The main contribution of this paper is a proposal of mathematical modelling of a hydrodynamically actuated mechanical system using numerical simulations of a system of two degrees of freedom. The structure of the paper is as follows: basic assumptions of the systems in this study and some background are given in Section 2. A numerical setup is presented in Section 3. Finally, Sections 4 and 5 present a discussion and conclusions of this work.

## 2. BACKGROUND

In this work, a hydrodynamically actuated mechanical system is under study. DRDC has been working with the Robotics and Mechanisms (RAM) Lab at UNB and other partners to develop a reliable method to dock UUVs to moving submarines or surface vessels. The proposal consists of attaching to the outer hull of the vessel a robotic arm which, using an array of sensors and actuators, will be able to autonomously capture a UUV. In [22] and [23], both undergraduate senior design projects, multiple UUV tracking manipulator were proposed, one of them being a serial manipulator with three degrees of freedom (three joints).

As in the wheel of inertia, cart-pendulum, or pendubot, this hydrodynamically actuated manipulator is an interesting test bed for analysis and control synthesis because it can be seen as two interacting dynamic systems: a fluid and a solid [24]. There are some features that make this manipulator an interesting test bed for analysis and control synthesis. First, the manipulator by itself could be analyzed as an inverted pendulum that moves with a force caused by a fluid [25]. Depending on the position of the frame, there will be gravity-bounded nonlinear terms affecting the behaviour of the system. Second, considering the effects of fluid on the solid, nonlinear quadratic dynamics will affect the mechanical system (*e.g.*, [5, 17]). Third, non-collocated dynamics will also affect the behaviour of the entire system because of its operation mode, which is very interesting for control design purposes.

In [26], the primary concern was to develop an arm capable of producing the dynamical motion for the capture of a UUV. An interesting design concept is that the first revolute joint (the one elevating the arm) can be either a) directly actuated (using a hydraulic actuator) [27] or b) indirectly actuated via “hydrodynamic action” produced by the interaction of the incoming water flow with the actively pitching wing placed at the end of the arm [1]. In the prototype, the wing with 2 m span and a cord of 0.6 m is mounted on a cylinder with 2 m stroke as shown in Figure 1. The current design, as it has been built, is schematically illustrated in Figure 2 (see [28]).

There are three operation modes for the manipulator: actuated, underactuated, and overactuated. Considering the underactuated mode, the manipulator will be actuated using the hydrodynamic force of the water through a wing-shaped link. Figure 2 shows a sketch of the robot, where $\theta_1$ is the angular position or elevation of arm, in the $x$-$z$ plane and $\theta_2$ denotes de angular position of the wing-shaped link in the $y$-$z$ plane. In

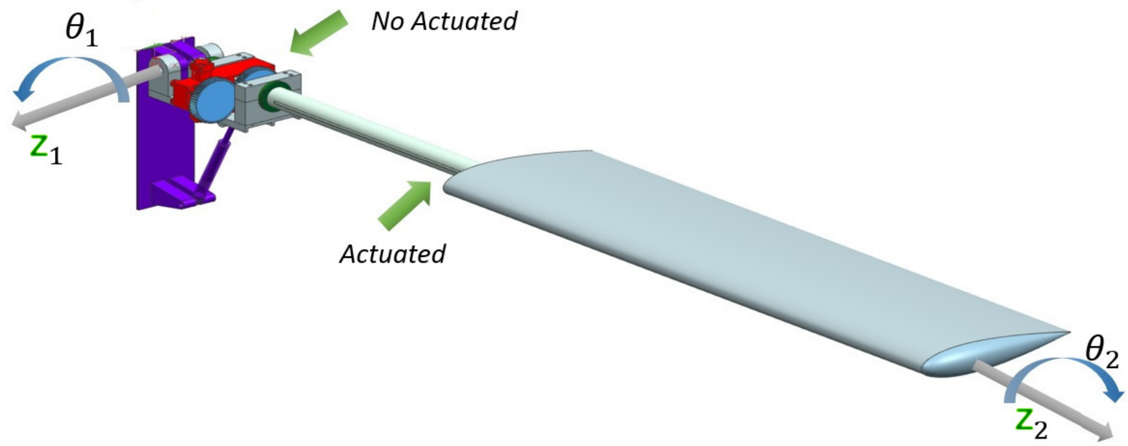

Fig. 2. The three-link manipulator.

this robot, the fluid affects the system in the *x*-*y* plane. Using the actuator to change the angle $\theta_2$, in other words, the angle of attack, it is possible to indirectly change the angular position $\theta_1$.

The results that are shown in this work belong to an early stage of this research. In order to present clearly the main idea of this work, instead of using a complex system such as the hydrodynamically actuated manipulator, a well-known LTI system is considered. Indeed, a mass-spring-damper is used as a test bed for the numerical experiments. So, the only relation between these two systems is that both are mechanical systems affected by viscous friction and external perturbations, such as the hydrodynamics.

The main idea of this work is to show a comparison between three numerical experiments: an MSD coupled with a PDE, and the same MSD affected by an approximation of the same PDE, and a nonlinear version of the MSD, considering added mass and drag friction. From a computational point of view, this work can be considered for a deeper understanding of the hydrodynamics and dynamics involved in this system. Moreover, it can help to visualize the performance of an underactuated and hydrodynamically actuated system with control design.

## 3. NUMERICAL SIMULATIONS

In this section, the main objective is to propose a mathematical model through numerical experiments using a mechanical system as a test bed. In order to support the theoretical results, an experimental design will be considered in future work. The proposed control design will be implemented in a real-time test bed, such as the underactuated prototype mentioned before. Through numerical simulations and experimental exercises, the theoretical and experimental results will be compared for validation. This will allow us to ensure a good performance of the mechatronic arm and the proposed control law in spite of uncertainties (external or parametric, among others).

In this work, a numerical experiment is shown using an MSD affected by a travelling wave. Numerical simulations are made using SIMULINK and MATLAB. First, an MSD is coupled with a first order partial differential equation: linear advection equation (PDE). Second, the same system is forced with an approximation of the same PDE, and last, a nonlinear version of an underwater MSD is considered (an MSD with added mass and drag friction). The main idea of this comparison is to see how the same physical system could have different representations. Since there are no experimental simulations of this mechanical system, the criteria of comparison are qualitative ones: for instance, the computational time, the complexity of the calculations, nonlinear terms involved, among others.

A mathematical model of an MSD system, affected by viscous friction and by an external perturbation is

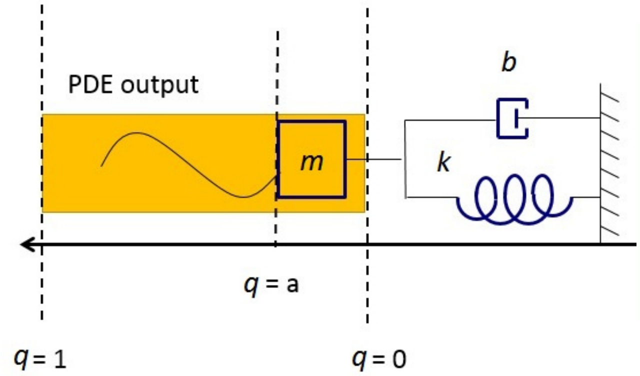


Fig. 3. The MSD system.

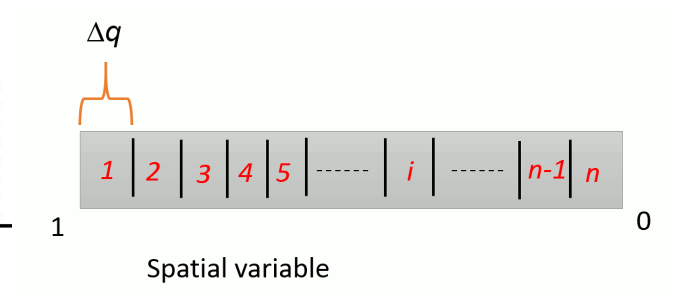


Fig. 4. Method of Lines.

considered to illustrate the qualitative behaviour of the models. Figure 3 shows the test bed consisting of an MSD system coupled with a PDE modelled by

$$\begin{aligned} m\ddot{q} &= -kq - F(\dot{q}) + hu(q,t), \\ \frac{\partial u}{\partial t} &= -\upsilon\frac{\partial u}{\partial q}. \end{aligned} \tag{1}$$

where $m$ denotes the mass, $q$ denotes the position of the mass with respect to the origin, $k$ is the spring constant, $F(\dot{q})$ denotes the friction of the system that will be described further, $h$ is a constant of proportionality to describe the relation of the force caused by the travelling wave and its position, and with the initial conditions:

$$\begin{aligned} u(1,t) &= U(t), \\ u(0,t) &= 0, \end{aligned} \tag{2}$$

where $u(1,t)$ denotes the solution of the PDE at the position $q = 1$. In 1939, J.M. Burgers simplified the Navier-Stokes equations. Moreover, let us consider an incompressible fluid, laminar flow and with no rotational forces. Then, the one-dimensional Burger's equation without forcing is given by

$$\frac{\partial u}{\partial t} + \upsilon\frac{\partial u}{\partial q} = \nu\frac{\partial^2 u}{\partial q^2}. \tag{3}$$

When $\nu = 0$ and $\upsilon$ is a constant representing the advection velocity, the Burger's equation becomes the linear advection equation:

$$\frac{\partial u}{\partial t} = -\upsilon\frac{\partial u}{\partial q}. \tag{4}$$

As was mentioned above, the solution of the equation (4) can be constructed by the Method of Characteristics, or the Method of Lines (MOL) [29], among others.

The friction force $F(\dot{q})$ is simply described by a linear term

$$F(\dot{q}) = b\dot{q}. \tag{5}$$

where $b$ denotes the level of viscous friction.

MOL methodology replaces the problem of PDEs with a set of linear ODEs. Thus, a model consisting of systems of a coupled PDE-ODE can be numerically solved with this methodology. In other words, the MOL is a flexible procedure to implement more complex dynamics in a numerical platform. Here, MOL will be considered in order to discretize the infinite dimensional part of this system, *i.e.*, in order to implement this coupled PDE-ODE numerically, finite differences methodology with Euler method is considered, in other words, the derivative of $u(q,t)$ with respect to $q$ will be estimated using the value of $u(q,t)$ with respect to the spatial variable $x$ divided in a finite number of $i$ slots :

$$u_q(q,t) \sim \frac{u^i - u^{i-1}}{\delta q}, \tag{6}$$

where the subindex $i = 1, 2, \ldots, n$ denotes the slot, part of the spatial variable, $n$ is the number of grid points, according to MOL (Figure 4). Then, $u^i(t)$ denotes the solution $u(t)$ with respect to the slot $i$.

Using equation (6) in the second part of equation (1),

$$\frac{du^i}{dt}(t) + \upsilon \frac{u^i - u^{i-1}}{\delta q} = 0, \tag{7}$$

with initial conditions

$$\begin{aligned} u(1,t) &= U(t), \\ u_1 &= U(q(i)) = g(t), \end{aligned} \tag{8}$$

where $g(t)$ denotes the solution of equation (1) when the position is zero.

Note that using MOL, instead of a coupled PDE-ODE, the discretization algorithm gives an $n-1$ order linear system, and $\delta q$ is the spacing between each grid. The state-space representation of the system in (1), after the spatial discretization, is

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \frac{1}{m}\Big(-kx - by + hu^i(t)\Big), \\ \dot{u} &= -\upsilon \frac{u^i - u^{i-1}}{\delta q}, \end{aligned} \tag{9}$$

where the state vector $[x(t) \quad y(t) \quad u(t)]^T = [q \quad \dot{q} \quad u]^T$.

The result of this procedure is an LTI (Linear Time-Invariant) dynamic system of order $n+1$, in other words, an $n+1$ set of LTI ODEs that can be numerically solved using an integration method such as Euler or Runge-Kutta. Note that this is a set with $n+1$ equations due to the dynamics of the discretized PDE $(n-1)$ plus the MSD dynamics (which itself is a second order system). In this work, the Runge-Kutta method is considered. Moreover, in order to implement the PDE with MOL methodology, the initial condition of the PDE is considered as the input of the discrete system, and the output of the discrete system will depend on the position of the MSD system. In this numerical experiment, as Figure 3 shows, the MSD has an equilibrium point in $x = 0.5$ and is inside the PDE domain.

Now, for the second experiment, instead of using a PDE output to model a travelling wave, the mathematical approximation $h(x,t) = A\sin(x - \omega t)$ will be used as a solution of the PDE. That is, the representation of the travelling wave, where $A$ is the amplitude of the wave or signal and $\omega$ is its frequency. Moreover, this equation is the input of an MSD system, as follows

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \frac{1}{m}\Big(-kx - by + cA\sin(x - \omega t)\Big). \end{aligned} \tag{10}$$

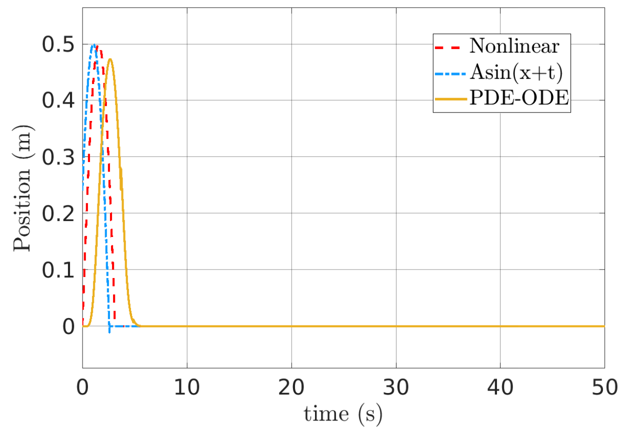


Fig. 5. The input of the MSD system.

In this case, notice that also $h(x,t) = cA\sin(x - \omega t)$ depends on the position of the MSD system and $c$ is a constant of proportionality that describes the relation between a force caused by the travelling wave and its position.

In [2], an underwater manipulator is modelled using added mass and drag coefficients for a one-link manipulator. In this work, the added mass and drag coefficient are considered for an MSD system. This mechanical system is represented by a nonlinear ODE, as follows:

$$(m + m_a)\ddot{q} = -kq - F(\dot{q}) + u(q,t) - \gamma(\dot{q})^2 \operatorname{sgn}(\dot{q}), \tag{11}$$

where $m_a$ is the added mass and $\gamma$ is the level of drag friction. In the first comparison, the level of drag friction is set to zero, in order to focus on the added mass dynamics (see Figures 6 through 9).

The idea is to compare the three numerical setups for an MSD system: a coupled PDE-ODE system, an MSD with a travelling wave as an input, and a nonlinear version of an underwater MSD. Parameters of an MSD system are considered as follows: the mass of the pendulum is $m = 1$ kg, the stiffness coefficient $k = 1$ N/m. The viscous friction is given by $b = 1$ kg/s. The advection velocity $\upsilon = 0.5$ m/s is considered. The constants $A = 0.5$ m and $\omega = 1$ Hz were considered in order to implement $h(x,t) = cA\sin(x - \omega t)$ in MATLAB. The constants of proportionality are fixed as $h = 1$ N/m and $c = 1$ N/m. The added mass is $m_a = 1$ kg and the drag force is equal to zero. In the third numerical simulation, the drag force is $d_a = 0.2$ kg/m (see Figures 10 and 11). In order to implement MOL methodology, the spatial variable is split as $n = 20$. The initial conditions for the MSD system, selected for all experiments, are fixed as $x(0) = 0.5$ m and $y(0) = 0$ m/s for the position and velocity, respectively.

Note that a difference between these numerical experiments is the type of equation to describe the dynamics of the system. The coupled PDE-ODE, after the MOL procedure, becomes a system of Differential Algebraic Equations (DAE) because of the implementation of the initial conditions. The second modelling using the explicit solutions of the PDE becomes a continuous dynamic system. Moreover, since the explicit solution of the PDE depends on time, it becomes an autonomous dynamic system. The third type of modelling is a nonlinear system since drag friction is also considered. If the level of drag friction is set to zero, the MSD system becomes a Linear Time-Invariant System.

## 4. DISCUSSION

A comparison of three models for an underwater mass-spring-damper is shown. Numerical simulations were shown in order to compare the qualitative behaviour of different representations of the same hydrodynamically actuated mechanical system. A coupled PDE-ODE, an MSD with an autonomous input, and

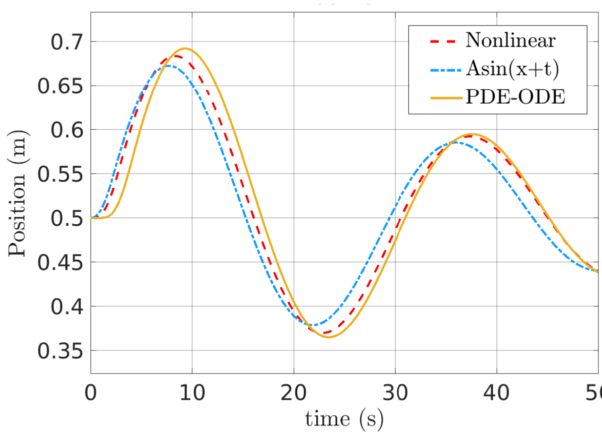


Fig. 6. The position of the MSD system.

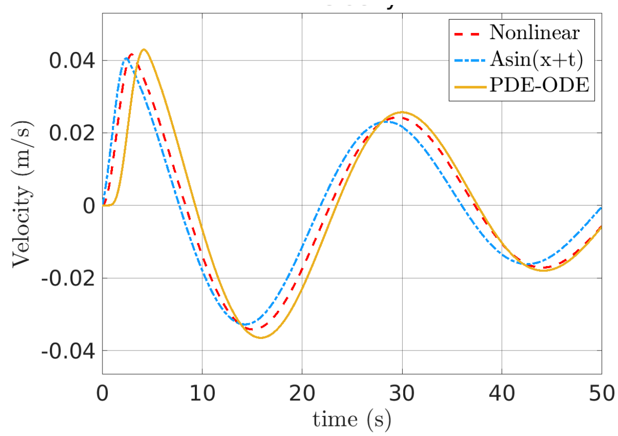


Fig. 7. The velocity of the MSD system.

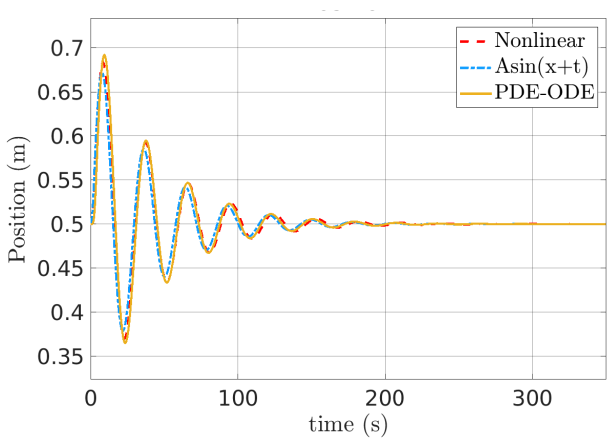


Fig. 8. The position of the MSD system: the steady state.

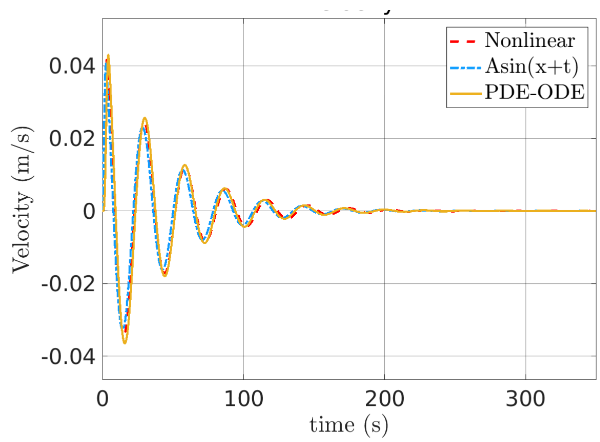


Fig. 9. The velocity of the MSD system: the steady state.

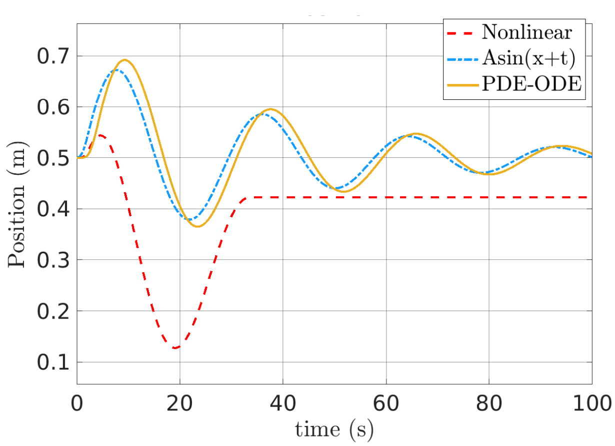


Fig. 10. The position of the MSD system: the drag friction included.

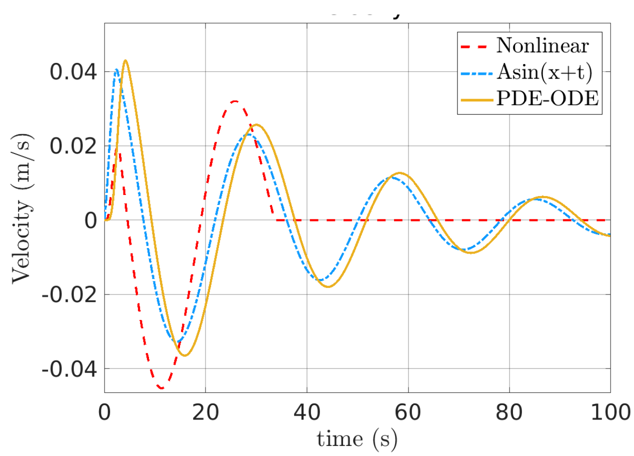


Fig. 11. The velocity of the MSD system: the drag friction included.

nonlinear underwater MSD were presented. All of them were implemented on the same platform, with the similar parameters. Figure 5 shows the input signal used in all systems. Figures 6 and 7 show the transient response of the MSD system. It can be seen that the nonlinear representation and MSD with $A\sin(x+\omega t)$

as input, have similar behaviour: as a smooth nonlinear function. However, the velocity of the coupled PDE-ODE has a transient response more like a sine function. The nonlinear effect in the coupled PDE-ODE looks like diminished and it could be caused by the MOL discretization process, since the spatial state variable of the PDE is filtered. Figures 8 and 9 show the steady state response of the MSD system. All three representations are very similar in steady-state response. Figures 10 and 11 show the nonlinear MSD system with drag friction. The behaviour of this nonlinear term is very different from the other representations, even at small scales.

Considering this numerical experiment, it is possible to see some advantages and disadvantages of each modelling method. Focusing on numerical simulations, the coupled PDE-ODE model could give better results considering that the fluid-solid interaction could be described completely. A drawback could be in the computational effort and time-consuming requirements. If the main idea is to work on a control design, the nonlinear model and the MSD with an autonomous input come to be more suitable. The structure of the equation is easier to manage and simulate than a coupled PDE-ODE. However, the added mass is a parameter using an offline process (CFD, analytical approximations, among others), in other words, this parameter has a percentage of error, thus the control design must have a high level of robustness and adaptation to parametric uncertainties and no modelled dynamics (high gain control strategy, for instance). Consequently, the control design comes to be more difficult for the implementation in an experimental setup. Considering a specific objective one or other model could be in consideration. Another main difference between the nonlinear model and the MSD with an autonomous input is precisely the non-autonomous/autonomous model of the same underwater mechanical system. For control design purposes, this is a very important feature to consider. There are techniques for autonomous systems that not necessarily satisfies the same requirements for non-autonomous systems.

## 5. CONCLUSION

A mathematical model of a hydrodynamically actuated mechanical system was under study. With this aim, numerical experiments were shown in order to compare three mathematical representations of the same system. The PDE-ODE representation requires more experience in dynamics and programming. Moreover, SIMULINK is a platform not suitable to implement directly this kind of dynamic system, so the Method of Lines was considered in order to do the implementation. The MSD with a sine function as input and the nonlinear MSD model are more suitable for a platform such as MATLAB, and a much simpler for implementation purposes. For future work, this result can be easily generalized for the multidimensional case. Moreover, it can be extended that when the coupling of the PDE-ODE is more complex, the output of the ODE affects the results of the PDE and also a more complex version of the PDE could be considered, such as the inviscid Burger's equation or the full Burger's equation.

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