INTEREVAL OBSERVER DESIGN FOR LINEAR CONTINUOUS TIME-VARYING DELAY SYSTEMS: NEW RESULTS

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ABSTRACT

This paper is concerned with the design of interval observer for linear continous time-delay systems. By a coordinates transformation, an interval state observer is designed. Both matrices observer gain, the lower and the upper, are obtained by solving a Sylvester's matrix equation. An implementation of the interval observer is proposed. Numerical example is given to illustrate this approach.

Keywords: Interval observers; positive time-varying delay systems; Sylvester's matrix equation.

DESIGN D'OBSERVATEUR INTERVALLE POUR LES SYSTÈMES LINÈAIRES CONTINUS À RETARD, VARIABLE DANS LE TEMPS: NOUVEAUX RESULTATS

RÉSUMÉ

Des conditions d'existence d'observateur intervalle associé au système à retard variable dans le temps sont données. Une technique de synthèse des deux gains d'observateurs intervalles majorant et minorant en résolvant un système d'équations matricielle de Sylvester a été proposée. Des conditions assurant la positivité et la stabilité d'erreurs d'observation intervalle sont données. Un algoritme calculant les états estimés de sytème linèaire continu à retard, variable dans le temps a été donné. Pour la simulation de notre approche, un exemple numérique est proposé.

Mots-clés : observateur intervalle, les sytèmes linèaires continus à retard positifs ; système d'équations matricielle de Sylvester.

1. INTRODUCTION

Time-varying delay systems constitute a special class of dynamical systems, they are abundant in nature, frequently encountered in mechanic, robotic, engineering, chemical and economic systems, see for example, [1, 2]. Time-varying delay is often lead to instability or performance degradation of corresponding systems. For this reason, in the last few years a considerable attention has been devoted to the time varying delay systems. Most of the works related to positive invariance concept have been developed for time delay systems with constrained control. In [3, 4], the regulator problem of continuous time delay system with symmetric and non-symmetric constrained control has been formulated and solved, where the synthesis of state feedback has been handled.

Since the state of the system is often not fully available for measurement, output feedback has to be pursued, which is the general standard case in most practical problems. Thus, many works found in the literature have dealt with the output feedback constrained control [5, 6]. The problem of observer design for delayed systems is rather complex especially when the delay is time-varying [23, 24]. It is worth stressing that this problem has been recognized as a hard problem and is still one the open problems in control theory [7, 8].

Motivated by the above discussion, we deal with the problem of the estimation of unmeasured variables for linear continuous time-varying delay systems. The interval observer approach is recently developed and applied in many works, see for example [9–13], and [14]. This technique evaluates at any time lower and upper estimates of the system state, provided that bounds on the initial conditions are known [11]. With those properties, the closed-loop stabilization can be guaranteed [15]. The major aim of this paper is to handle the problem of the existence of interval observer for time-varying delay systems and to develop an algorithm to calculate the upper and the lower observers gain.

This paper is organized as follows. The problem statement and some preliminaries are given in Section II. The interval observer for time-varying delay systems is formulated in Section III with an algorithm to compute the upper and lower observer gains, which is based on a system of Sylvester matrix equation. The model implementation of interval observer is proposed in Section IV. Finally, a numerical example is given in Section V to demonstrate the effectiveness of the proposed approach.

Notation: Let M and N be two matrices of $\mathbb{R}^{n\times n}$ and x, y two vectors of $\mathbb{R}^n$. Then: M^+ (respectively M^-) is the matrix whose components are given by $M^+_{ij} = max(M_{ij},\, 0)$ (respectively $M^-_{ij} = max(-M_{ij},\, 0)$). Moreover, $|M|$ is the matrix, whose components are given by $|M_{ij}|$, for $i,\ j = 1,\dots,n$, and $\lambda_i(M)$ is an eigenvalue of M. $\mathbb{I}_n$ is the identity matrix of $\mathbb{R}^{n\times n}$. When M and N are symmetric, the notation $M \succeq N$ (resp., $M \succ N$), means that $M-N$ is positive semidefinite (resp., positive definite). The symbol $\star$ stands for symmetric block in matrix inequalities. M^T and $diag(M)$ denote respectively, the transpose and the diagonal of M. $x \le y$ if $x_i \le y_i$, for $i = 1,\dots,n$, ∂D is the boundary of the set D and $IntD$ is the interior of the set D. $\mathscr{C}^n_{h_m} = \mathscr{C}([-h_m,\, 0],\, \mathbb{R}^n)$ is the set of continuous maps from $[-h_m,\, 0]$ into $\mathbb{R}^n$.

2. PROBLEM STATEMENT

Consider the following state-space linear system with time-varying delays:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h(t)) + Bu(t), \\ \\ y(t) = Cx(t), \end{cases} \qquad \text{t>0}, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ are, respectively the system state, the input and the measured output. Moreover $A, A_d \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ are known real constant matrices and $h(t)$ is the time-varying delay.

The initial conditions of system Eq. (1) is given by:

$$x(t_0+\theta)=\phi(\theta) \text{ for } \theta \in [-h_m,\, 0], \text{ with } h(t) \leq h_{max} \tag{2}$$

where $\phi(.) \in \mathscr{C}^n_{h_m}$ is a vector of differentiable functions of initial values. For the sake of simplicity, $x(t_0+\theta)$ will be denoted as x_0 in the sequel, and we assume that the set of admissible initial conditions of system Eq. (1) is bounded and known such that

$$x_{0_{inf}} \leq x_0 \leq x_{0_{sup}} \tag{3}$$

where $x_{0_{inf}}, x_{0_{sup}} \in \mathscr{C}^n_{h_m}$.
The following assumption is needed throughout the paper.

Assumption 1. *The pair $(A+A_d,\, B)$ is controlable and the pair $(A+A_d,\, C)$ is observable.*

Assumption 1, which is equivalent to the stabilization of the system Eqs. (1–3) without delay, is a necessary condition for the existence of a linear stabilizing dynamic output feedback control law for the system Eqs. (1–3).

In this paper, we present a technique for the dynamic estimation of bounds on unmesasured variables of linear continuous time-varying delay systems Eqs. (1–3). Now, we recall some standard definitions of delayed systems, which will be useful the foregoing analysis and construction of the interval observer for system Eqs. (1–3)(see for example [5, 13, 16, 17]).

Definition 1. *A real square matrix A is called a positive matrix if all its elements are nonnegative, that is $A_{ij} \geq 0$.*

Definition 2. *A real square matrix A is called a Metzler matrix if all its off-diagonal elements are nonnegative, that is $A_{ij} \geq 0$, for $i \neq j$.*

Definition 3. *Consider the following delay system:*

$$\dot{\chi}(t) = f(t,\, \chi(t),\, \chi(t-h(t))), \tag{4}$$

with an initial condition $\phi(\theta) \in \mathscr{C}^n_{h_m}$. The system Eq. (4) is said to be positive if for every initial condition $\phi(\theta) \geq 0$, $\theta \in [-h_m\; 0]$, the solution $\chi(t)$ of Eq. (4) is positive, for all $t \geq 0$.

Definition 4. *An interval observer for system Eqs. (1–3) is a pair of lower and upper estimation $\{x_{sup},\, x_{inf}\}$ of the state $x(t)$ such that*

$$x_{inf}(t) \leq x(t) \leq x_{sup}(t) \quad \textit{for all } t > 0. \tag{5}$$

3. INTERVAL OBSERVER DESIGN

In this section, an interval observer is designed for time-varying delay system based on coordinate transformation of system Eqs. (1–3). It has been shown in recent works for linear and nonlinear systems without delay [11, 12] that by introducing a changes of variables, the interval observer can be developed and guaranteed the positivity of the upper and lower observations errors. A full interval observer is developed of this paper. It is noteworthy that this result can be easily extended to reduced-order interval observer. For system Eq. (1), there exist a non-singular matrix $M \in \mathbb{R}^{n\times n}$ such that

$$z = M^{-1}x, \tag{6}$$

Then from Eq. (6) the interval observer for system Eqs. (1–3) is designed as follows

$$\begin{cases} \dot{z}_{sup}(t) = (\widetilde{A} - \widetilde{\overline{L}}\widetilde{C}) z_{sup}(t) + \widetilde{A_d} z_{sup}(t-h(t)) + \widetilde{B}u(t) + \widetilde{\overline{L}}\widetilde{C} z_{sup}(t) \\ \dot{z}_{inf}(t) = (\widetilde{A} - \widetilde{\underline{L}}\widetilde{C}) z_{inf}(t) + \widetilde{A_d} z_{inf}(t-h(t)) + \widetilde{B}u(t) + \widetilde{\underline{L}}C z_{inf}(t), \end{cases} \tag{7}$$

where $\widetilde{A} = M^{-1}AM$, $\widetilde{A_d} = M^{-1}A_d M$, $\widetilde{B} = M^{-1}B$, $\widetilde{C} = CM$, $\widetilde{\overline{L}} = M^{-1}\overline{L}$ and $\widetilde{\underline{L}} = M^{-1}\underline{L}$, with $\overline{L}$, $\underline{L} \in \mathbb{R}^{n\times p}$ are the observed gains to be determined. The initial conditions of Eq. (7) is given by $z_{sup}(t+\theta) \in \mathscr{C}^n_{h_m}$ and $z_{inf}(t+\theta) \in \mathscr{C}^n_{h_m}$, for all $\theta \in [-h_m\ 0]$, which are denoted, respectively, by $z_{0_{sup}}$ and $z_{0_{inf}}$.

Remark 5. *It is noteworthy that in the new coordinates defined above, the original system Eq.* (1) *can be presented as follows*

$$\dot{z}(t) = \widetilde{A}z(t) + \widetilde{A_d}z(t-h(t)) + \widetilde{B}u(t),\ y(t) = \widetilde{C}z(t),\ t > 0, \tag{8}$$

with the initial condition given by $z(t_0+\theta) \in \mathscr{C}^n_{h_m}$*, for all* $\theta \in [-h_m\ 0]$*, denoted by* z_0*.*

Proposition 6. *Assume that Eq.* (3) *and assumption 1 hold, and for all* z_0, $z_{0_{sup}}$, $z_{0_{inf}} \in \mathscr{C}^n_{h_m}$ *the constraint* $z_{0_{inf}} \leq z_0 \leq z_{0_{sup}}$ *is satisfied. If there exist* $\overline{L}$, $\underline{L} \in \mathbb{R}^{n\times p}$ *and a non-singular matrix* $M \in \mathbb{R}^{n\times n}$ *such that the following are true:*

1. $\widetilde{A_d}$ *is positive;*
2. $\widetilde{A} - \widetilde{\overline{L}}\widetilde{C}$ *and* $\widetilde{A} - \widetilde{\underline{L}}\widetilde{C}$ *are Metzler;*
3. $(\widetilde{A} - \widetilde{\overline{L}}\widetilde{C} + \widetilde{A_d})$ *and* $(\widetilde{A} - \widetilde{\underline{L}}\widetilde{C} + \widetilde{A_d})$ *are Hurwitz;*

then for any input $u \in \Omega \subset \mathbb{R}^m$*, we have* $z_{inf}(t) \leq z(t) \leq z_{sup}(t)$ *for* $t > 0$*, with each state* $z_{sup}(t)$ *and* $z_{inf}(t)$ *of observer Eq.* (7) *converges to* $z(t)$ *that is given by Eq.* (6)*, where* $x(t)$ *is solution of Eq.* (1)*. Furthermore, the interval estimation of* $x(t)$ *is given by:*

$$\begin{cases} x_{sup}(t) = M^+ z_{sup}(t) - M^- z_{inf}(t) \\ x_{inf}(t) = M^+ z_{inf}(t) - M^- z_{sup}(t), \end{cases} \tag{9}$$

Proof. Consider the upper and lower observation errors of Eq. (7) defined by

$$e_{sup}(t) = z_{sup}(t) - z(t), \tag{10a}$$

$$e_{inf}(t) = z(t) - z_{inf}(t). \tag{10b}$$

then it follows from Eq. (8) and Eq. (7) that the augmented dynamic error system can be described by

$$\dot{e}(t) = \begin{pmatrix} \widetilde{A} - \widetilde{\overline{L}}\widetilde{C} & 0 \\ 0 & \widetilde{A} - \widetilde{\underline{L}}\widetilde{C} \end{pmatrix} e(t) + \begin{pmatrix} \widetilde{A_d} & 0 \\ 0 & \widetilde{A_d} \end{pmatrix} e(t-h(t)), \tag{11a}$$

where

$$e(t) = \begin{pmatrix} e_{sup}(t) \\ e_{inf}(t) \end{pmatrix}. \tag{11b}$$

Since, for any $\overline{L}$, $\underline{L} \in \mathbb{R}^{n\times p}$ and non-singular matrix $M \in \mathbb{R}^{n\times n}$ such that $\widetilde{A} - \widetilde{\overline{L}}\widetilde{C}$, $\widetilde{A} - \widetilde{\underline{L}}\widetilde{C}$ are Metzler and $\widetilde{A_d}$ is positive together with the constraint $z_{0_{inf}} \leq z_0 \leq z_{0_{sup}}$, which implies $e_0 \geq 0$ $\big($ i.e. $e(t_0+\theta) \geq 0$ for $\theta \in [-h_m,\ 0]\big)$, it follows that from [18], system Eq. (11) is always positive. Namely for any nonnegative initial condition $e_0 \geq 0$, the solution $e(t) \geq 0$ for all $t > 0$, which is equivalent to $z_{inf}(t) \leq z(t) \leq z_{sup}(t)$ for $t > 0$. Note that as in [17], the system Eq. (11) is asymptotically stable if and only if $(\widetilde{A} - \widetilde{\overline{L}}\widetilde{C} + \widetilde{A_d})$ and $(\widetilde{A} - \widetilde{\underline{L}}\widetilde{C} + \widetilde{A_d})$ are Hurwitz with $\widetilde{A_d}$ is positive. This implies that $e(t)$ converges to zero, so we have $z_{sup}(t)$ tends $z(t)$, and $z_{inf}(t)$ tends $z(t)$ as $t \to \infty$. As in [11, 19], by observing that $M = M^+ - M^-$ and $|M| = M^+ + M^-$, and substituting Eq. (6) into $z_{inf}(t) \leq z(t) \leq z_{sup}(t)$ we obtain

$$M^+ z_{sup}(t) - M^- z_{inf}(t) \leq x(t) \leq M^+ z_{inf}(t) - M^- z_{sup}(t). \tag{12}$$

From Eq. (12), we have Eq. (9), which completes the proof. □

Remark 7. *When matrix $\widetilde{A_d}$ is positive, the following hold.*

- *If $\widetilde{A} - \widetilde{\overline{L}}\widetilde{C}$ and $\widetilde{A} - \widetilde{\underline{L}}\widetilde{C}$ are both Metzler, then $(\widetilde{A} - \widetilde{\overline{L}}\widetilde{C} + \widetilde{A_d})$ and $(\widetilde{A} - \widetilde{\underline{L}}\widetilde{C} + \widetilde{A_d})$ are also Metzler;*
- *As shown in [17], if $(\widetilde{A} - \widetilde{\overline{L}}\widetilde{C} + \widetilde{A_d})$ is Hurwitz, then there exist a positive vector $\rho \in \mathbb{R}^n$ such that $(\widetilde{A} - \widetilde{\overline{L}}\widetilde{C} + \widetilde{A_d})\rho < 0$. Since $\widetilde{A_d}$ is positive, therefore $(\widetilde{A} - \widetilde{\overline{L}}\widetilde{C})\rho < 0$ and hence, $(\widetilde{A} - \widetilde{\overline{L}}\widetilde{C})$ is Hurwitz. Similarly, it can also be shown that $(\widetilde{A} - \widetilde{\underline{L}}\widetilde{C})$ is Hurwitz;*
- *According to assumption 1 an appropriate selection of the matrices $\overline{L}$ and $\underline{L}$ can assign a real part negative eigenvalues to matrices $(A - \overline{L}C + A_d)$ and $(A - \underline{L}C + A_d)$, respectively.*

The key part of proposition 6 is to show that augmented observer error system Eq. (11) is positive and asymptotically stable simultaneously. A natural question is: *how to find a non singular matrix M, and a both gain matrices $(\overline{L}, \underline{L})$ guaranteeing the stability of $(\widetilde{A} - \widetilde{\overline{L}}\widetilde{C} + \widetilde{A_d})$ and $(\widetilde{A} - \widetilde{\underline{L}}\widetilde{C} + \widetilde{A_d})$ respectively, also send A_d to a positive matrix $\widetilde{A_d}$ and map $(A - \overline{L}C)$ and $(A - \underline{L}C)$ to Metzler matrices $(\widetilde{A} - \widetilde{\overline{L}}\widetilde{C})$ and $(\widetilde{A} - \widetilde{\underline{L}}\widetilde{C})$, respectively* ?. This problem has been recognized by several authors as a very challenging problem, and has not been fully resolved [13]. In this section, we show how this difficulty can be resolved by taking the same approach as in [14], where a Sylvester equation is resolved.

The main new feature in our method is based on the computation of both observe gain $\overline{L}$ and $\underline{L}$ matrices, unlike as in [14]. In order show our technique, let us use two Metzler and Hurwitz matrices $\overline{R}$ and $\underline{R}$, respectively, such that

$$\overline{R} = M^{-1}(A - \overline{L}C + A_d)M, \tag{13a}$$

$$\underline{R} = M^{-1}(A - \underline{L}C + A_d)M. \tag{13b}$$

It is straightforward to see that Eq. (13) can be rewritten as system of Sylvester matrix equation [20], that is if we consider $N = M^{-1}$, $\overline{Q} = M^{-1}\overline{L}$ and $\underline{Q} = M^{-1}\underline{L}$, then Eq. (12) becomes

$$\overline{R}N - N(A + A_d) + \overline{Q}C = 0, \tag{14a}$$

$$\underline{R}N - N(A + A_d) + \underline{Q}C = 0. \tag{14b}$$

This construction appears also in [14], but with only one equation. It amounts to the fact that, the authors have assumed in [14] that the upper observer gain $\overline{L}$ is equal to the lower observer gain $\underline{L}$, which is obviously very particular case. Here, we deal with both the upper and lower observer design, and the computation of observer gain matrices is implemented. Let us first observer that Eq. (14) is equivalent to

$$S \begin{pmatrix} \eta \\ \overline{q} \\ \underline{q} \end{pmatrix} = 0, \tag{15}$$

Algorithm 1 procedure to compute the observer gain matrices $\overline{L}$ and $\underline{L}$

Given two Metzler and Hurwitz matrices $\overline{R}$ and $\underline{R}$
Compute the parametric solution $(\eta \quad \overline{q} \quad \underline{q})$ of equation (15)
for a given vector $\alpha = \left(\alpha_1, \ldots, \alpha_l\right) \neq 0$ **do**
  $M^{-1} \leftarrow N$;
  $\overline{L} \leftarrow \overline{Q}$;
  $\underline{L} \leftarrow \underline{Q}$;
  Check the conditions of proposition 6
  if $\widetilde{A_d}$ is positive and both matrices $(\widetilde{A} - \widetilde{\overline{L}}\widetilde{C})$ and $(\widetilde{A} - \widetilde{\underline{L}}\widetilde{C})$ are Metzeler **then**
    return
    α
  else
    Change the vector α
  end if
end for

where $S = \begin{pmatrix} -\mathbb{I}_n \otimes A_0^T + \overline{R} \otimes \mathbb{I}_n & \mathbb{I}_n \otimes C^T & 0 \\ -\mathbb{I}_n \otimes A_0^T + \underline{R} \otimes \mathbb{I}_n & 0 & \mathbb{I}_n \otimes C^T \end{pmatrix}$, with $A_0 = A + A_d$, $\eta = vec(N)$, $\overline{q} = vec(\overline{Q})$, and $\underline{q} = vec(\underline{Q})$. Let $\otimes$ denotes the Kronecker product and $vec(.)$ is vector valued function of any matrix $(.)$, that is $\eta = \left(N_{11}, \ldots, N_{1n}, \ldots, N_{nn}\right)^T$, $\overline{q} = \left(\overline{q}_{11}, \ldots, \overline{q}_{1n}, \ldots, \overline{q}_{nn}\right)^T$, and $\underline{q} = \left(\underline{q}_{11}, \ldots, \underline{q}_{1n}, \ldots, \underline{q}_{nn}\right)^T$. According to assumption 1 and from [21] and [11], we can solve equation Eq. (15) by finding a basis of $kerS$, the null subspace of the matrix S, such that for every element α_i of a real numbers, we have

$$(\eta \quad \overline{q} \quad \underline{q})^T = \sum_{i=1}^{l} \alpha_i \beta_i, \qquad l = dim(kerS), \tag{16}$$

where β_i is an element of the basis of $kerS$. Finally, it is obvious that the proposition 6 can supplied an algorithm 1 to compute effectively the observer gain matrices $\overline{L}$ and $\underline{L}$ and the non-singular matrix M that is related to coordinate transformation.

The result of proposition 6 means that the observer Eq .(7) is similar to the observer in the original coordinates defined by:

$$\dot{x}_{sup}(t) = (A - \overline{L}C)x_{sup}(t) + A_d x_{sup}(t - h(t)) + Bu(t) + \overline{L}Cx(t) \tag{17a}$$
$$\dot{x}_{inf}(t) = (A - \underline{L}C)x_{inf}(t) + A_d x_{inf}(t - h(t)) + Bu(t) + \underline{L}Cx(t), \tag{17b}$$

and if all their conditions are satisfied then under the relation Eq .(3), we have $x_{inf}(t) \leq x(t) \leq x_{sup}(t)$ for all $t > 0$. Note that the similar interval observer has been given in [22] in a different context.

Remark 8.

- *From Eq .(13) there are several degrees of freedom in the selection of $\overline{R}$ and $\underline{R}$ such that both are Metzler and Hurwitz matrices. This lends many possibilities to pick the desired pole associated to the interval observer;*

- *The initial conditions of interval observer Eq .(7) can be easily handled. In fact from Eq .(9), we obtain the following relation*

$$z_{0_{sup}} = |M^{-1}| x_{0_{sup}}, \quad with \quad z_{0_{inf}} = -z_{0_{sup}} \tag{18}$$

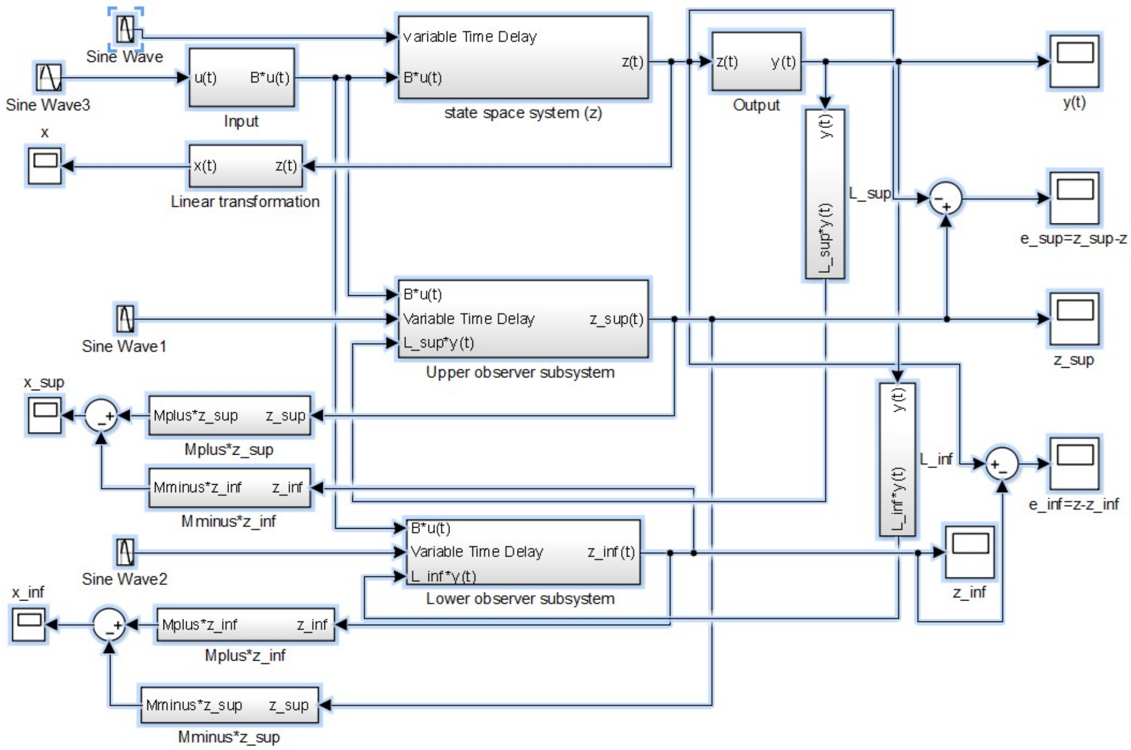

Fig. 1. Upper and lower observers gains implementation

4. INTERVAL OBSERVER ARCHITECTURE FOR TIME-VARYING DELAY SYSTEMS

The basic idea underlying interval observer designs for linear continous time-varying systems is to find a linear transformatiom such that, in the new coordinates of state space system (z) that has an asymptotically convergent interval observer(z_{sup}, z_{inf}) which is governed only by upper and lower observation errors(e_{sup}, e_{inf}). Note that the vector $\begin{pmatrix} u \\ y \end{pmatrix}$ can be seen as the input of the interval observer system Eq. (7), which can be rewritten as

$$\begin{cases} \dot{z}_{sup}(t) = (\widetilde{A} - \widetilde{\overline{L}}\widetilde{C}) z_{sup}(t) + \widetilde{A_d} z_{sup}(t-h(t)) + \begin{pmatrix} \widetilde{B} & \overline{L} \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} \\ \dot{z}_{inf}(t) = (\widetilde{A} - \underline{L}\widetilde{C}) z_{inf}(t) + \widetilde{A_d} z_{inf}(t-h(t)) + \begin{pmatrix} \widetilde{B} & \underline{L} \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} \end{cases} \tag{19}$$

It is noteworthy that, by using the last system Eq. (19), the implementation of the interval observer can be depicted in Fig. 1. The model depicted in Fig. 1 is very general. The upper observer gain $\overline{L}$ is not necessary equal to the lower observer gain $\underline{L}$.

5. EXAMPLE

Let consider the following system

$$\begin{cases} \dot{x}_1(t) = -4.2x_1(t) + x_2(t) + x_1(t-h(t)) + x_2(t-h(t)), \\ \dot{x}_2(t) = -x_2(t) + u(t), \\ y(t) = x_1(t) \end{cases} \tag{20}$$

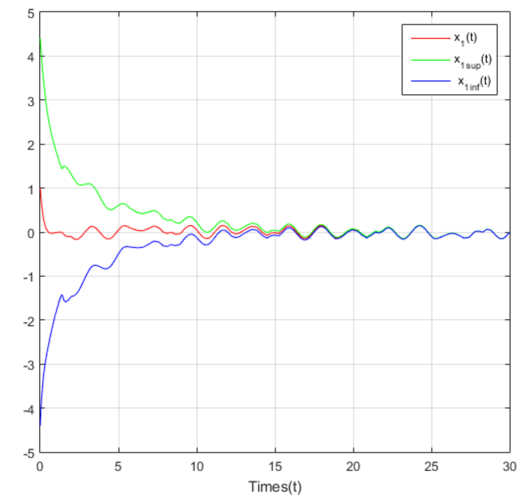

Fig. 2. Evolution of first state x_1 and its convergent intate observer.

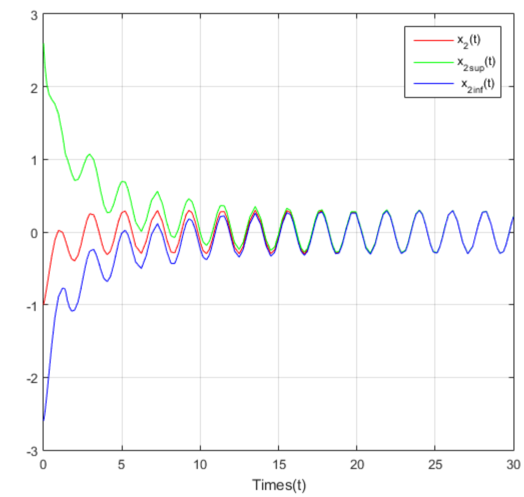

Fig. 3. Evolution of second state x_2 and its convergent interval observer.

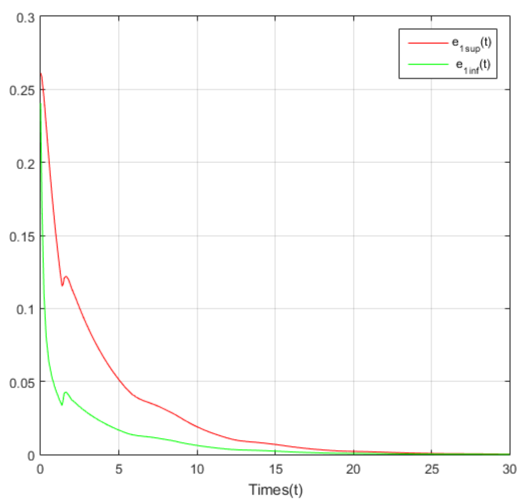

Fig. 4. Simulation of upper and lower observation errors.

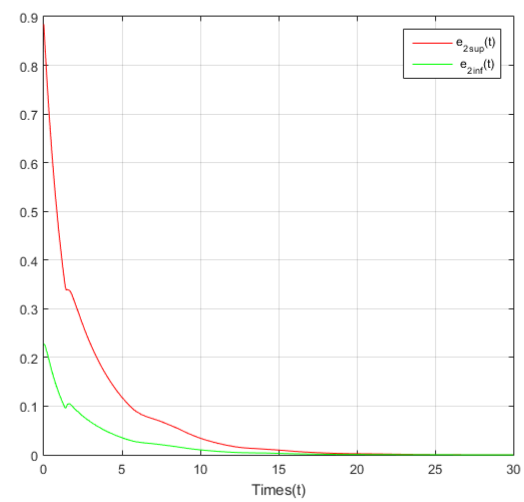

Fig. 5. Simulation of upper and lower observation errors.

with $u(t) = u_0 sin(3t)$ and $h(t) = h_m sin(0.95t+1)$, in this case $u_0 = 2$ and $h_m = 1.9$. We measure only the state x_2, we want to estimate state x_1. We assume also that the set of possible initial conditions is bounded and known such that for $\theta \in [-h_m\ 0]$, $\phi(\theta) \in \left\{\phi \in \mathscr{C}([-h_m,\ 0],\ \mathbb{R}^2) : |\phi_i| \leq 1,\ i = 1,2\right\}$. We apply algorithm 1 in order to design the interval observer $\overline{L}$ and $\underline{L}$ giving bounds on all states x of system described by Eq .20. By choosing the matrices $\overline{R}$ and $\underline{R}$ such that

$$\overline{R} = \begin{pmatrix} -0.95 & 2.1 \\ 1.7 & -5.49 \end{pmatrix}, \quad \underline{R} = \begin{pmatrix} -0.95 & 2.1 \\ 2.73 & -9.49 \end{pmatrix}, \tag{21}$$

and solving the Sylvester matrix equation Eq .(13), we obtain the transition matrix :

$$M = \begin{pmatrix} 2.9713 & -11.5383 \\ -3.1383 & 3.4070 \end{pmatrix}, \tag{22}$$

and

$$\overline{Q} = \begin{pmatrix} 0.5465 \\ -0.0533 \end{pmatrix}, \qquad \underline{Q} = \begin{pmatrix} 0.5465 \\ -0.3998 \end{pmatrix}. \tag{23}$$

This leads to

$$\overline{L} = \begin{pmatrix} 2.2388 \\ -1.8966 \end{pmatrix} \text{ and } \underline{L} = \begin{pmatrix} 6.2365 \\ -3.0770 \end{pmatrix}. \tag{24}$$

The following matrices

$$\widetilde{A} - \widetilde{\overline{L}}\widetilde{C} = \begin{pmatrix} -0.9717 & 1.0373 \\ 1.6799 & -6.4682 \end{pmatrix}, \widetilde{A_d} = \begin{pmatrix} 0.0217 & 1.0626 \\ 0.02 & 0.9782 \end{pmatrix} \text{ and } \widetilde{A} - \widetilde{\underline{L}}\widetilde{C} = \begin{pmatrix} -0.9717 & 1.0373 \\ 2.7099 & -10.4682 \end{pmatrix}, \tag{25}$$

satisfy, respectively all conditions of Proposition 6. In this case, $dim(KerS) = 1$ and $\alpha_1 = 1$, where the real number α_1 can be chosen arbitrary. The simulation results are shown in Fig. 2 and Fig. 3. As shown in Fig. 4 and Fig. 5, the upper error $e_{sup}(t) = z_{sup}(t) - z(t)$ and lower error $e_{inf}(t) = z(t) - z_{inf}(t)$ are always nonnegative and asymptotically stable.

6. CONCLUSION

We have provided constructive approach for estimation problem for linear continuous time-varying delay systems. An algorithm to calculate the upper and the lower observers gain is developed. The procedure is based on coordinate transformation of original system. The corresponding technique of the observer design requires solving of the Silvester's equation.

REFERENCES

1. J. P. Richard, "Time-delay systems: an overview of some recent advances and open problems," *Automatica*, vol. 39, no. 10, pp. 1667–1694, 2003.
2. M. Wu, Y. He, and J.-H. She, *Stability Analysis and Robust Control of Time-Delay Systems*. Beijing and Springer-Verlag Berlin Heidelberg, 2010.
3. A. Hmamed, A. Benzaouia, and H. Bensalah, "Regulator problem for linear continuous-time delay systems with nonsymmetrical constrained control," *IEEE Trans. Autom. Control*, vol. 40, no. 9, pp. 1615–1619, 1995.
4. J. C. Hennet and S. Tarbouriech, "Stability and stabilization of delay differential systems," *Automatica*, vol. 33, no. 3, pp. 347–354, 1997.
5. A. Benzaouia, A. Hmamed, and F. Tadeo, "Stabilisation of controlled positive delayed continuous-time systems," *International Journal of Systems Science*, vol. 41, no. 12, pp. 1473–1479, 2010.
6. S. Xu, G. Feng, Y. Zou, and J. Huang, "Robust controller design of uncertain discrete time-delay systems with input saturation and disturbances," *IEEE Trans. Autom. Control*, vol. 57, pp. 2604–2609, 2012.
7. F. Blanchini and S. Miani, *Set-Theoretic Methods in Control*. Birkhauser Boston. Basel. Berlin, 2008.
8. A. Benzaouia, *Saturated Switching Systems*. Springer-Verlag, 2012.
9. J. Gouzé, A. Rapaport, and M. Hadj-sadok, "Interval observers for uncertain biological systems," *Ecological Modelling*, vol. 133, pp. 45–46, 2000.
10. A. Rapaport and J. L. Gouzé, "Parallelotopic and practical observers for non-linear uncertain systems," *International Journal of Control*, vol. 76, pp. 237–251, 2003.

11. R. Tarek, E. Denis, and A. Zolghadri, “Interval state estimation for a class of nonlinear systems,” *IEEE Trans. Autom. Control*, vol. 57, pp. 260–65, 2012.
12. F. Mazenc and O. Bernard, “Interval observers for linear time-invariant systems with disturbances,” *Automatica*, vol. 47, pp. 140–147, 2011.
13. D. Efimov., W. Perruquetti., and J.-P. Richard, “Interval estimation for uncertain systems with time-varying delays,” *International Journal of Control*, vol. 86, p. 1777, 2013.
14. A. Polyakov, D. Efimov, W. Perruquetti, and J.-P. Richard, “Output stabilization of time-varying input delay systems using interval observation technique,” *Automatica*, vol. 49, pp. 3402–3410, 2013.
15. A. Rapaport and J. Harmand, “Robust regulation of a class of partially observed nonlinear continuous bioreactors,” *Journal of Process Control*, vol. 12, pp. 291–302, 2002.
16. T. Kaczorek, “A realization problem for positive continuous-time systems with reduced numbres of delyas,” *Int. J. Appl. Math. Comput. Sci.*, vol. 16, no. 3, pp. 325–331, 2006.
17. H. U. Ait-Rami, M. and Tadeo.F, “Positive observation problem for linear time-delay positive systems,” in *Mediterranean Conference on Control and Automation, July 27-29, Athens - Greece*, 2007.
18. W. M. Haddad and V. Chelleboina, “Stability theory for nonnegative and compartmental dynamical systems with delay,” *Systems and Control Letters*, vol. 51, pp. 355–361, 2004.
19. H. Bensalah and L. Baron, “Positive Invariance of constrained linear continuous-time delay system with delay dependence.” In “Proceeding of International Conference of Control, Dynamic Systems, and Robotics,” Paper No. 179. Ottawa, Ontario, Canada, May 7-8 2015.
20. H.-S. Q. Yu-Peng Qiao and D.-Z. Cheng, “Parameterized solution to a class of syslvester matrix equations,” *International Journal of Automation and Computing*, vol. 7, pp. 479–483, 2010.
21. H. Neudecker, “A note on kronecker matrix products and matrix equation systems,” *SIAM J. Appl. Math.*, vol. 17, no. 3, pp. 603–606, May 1969.
22. D. Efimov., A. Polyakov., and J.-P. Richard, “Interval estimation for systems with time delays and algebraic constraints,” in *European Control Conference (ECC) June 24-27. Strasbourg, France*, 2014, pp. 678–683.
23. M. Darouach, “Linear function observers for systems with delays in state variables,” in *IEEE Trans. Autom. Control*, vol. 46, no.3, pp. 491–496, 2001.
24. F. Mazenc, S. Niculescu, and O. Bernard, “Exponentially stable interval observers for linear systems with delay,” *SIAM J. Control Optim*, vol. 15, no. 1, pp. 286–305, 2012.