THE KINEMATICS Of MOBILE ROBOTS WITH ORIENTABLE SINGLE AND DUAL WHEELS ROLLING ON A PLANE

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ABSTRACT

The kinematics of wheeled mobile robots (WMRs) equipped with an arbitrary number of conventional orientable wheels or dual-wheel units (DWUs) is the subject of this paper. The configuration of a robot with s such wheels is defined by $3+2s$ independent variables, three for the platform configuration and two for each single wheel or dual-wheel unit. However, not all variables related to the wheel configuration are independent since, for the platform to be able to undergo an arbitrary planar motion, the wheel-unit configurations must satisfy $s-2$ *compatibility* conditions. Hence, the degree of freedom (dof) of the system is $3+2s-(s-2)=s+5$. It is made apparent that wheeled mobile robots equipped with at least three orientable wheels can be made omnidirectional.

La cinématique de robots mobiles munis de roues orientables simples et duales roulant sur un plan

Cet article traite de la cinématique de robots mobiles à roues (RMRs) munis d'un nombre arbitraire de roues orientables ou d'unités de roues duales (URDs). La configuration d'un robot ayant s roues est définie par $3+2s$ variables indépendantes, trois pour la configuration de la plate-forme et deux pour chaque roue simple ou, le cas échéant, pour chaque URD. Toutefois, les variables relatives à la configuration des roues ne sont pas toutes indépendantes puisqu'une unité de roue doit satisfaire à $s-2$ conditions de *compatibilité*, afin que la plate-forme puisse effectuer un mouvement planaire arbitraire. Par la suite le degré de liberté de ce système est $3+2s-(s-2)=s+5$. Ainsi les RMRs munis d'au moins trois roues orientables peuvent devenir omni-directioneles.

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1 Introduction

Wheeled mobile robots (WMRs) are still the most commonly used mobile robots. The kinematics of different types of these robots has been widely discussed; in our opinion, however, some questions still are not clear, e.g., the mobility of WMRs with more than three orientable wheels or dual-wheel units and the possible actuation schemes. A dual-wheel unit (DWU) comprises two wheels mounted coaxially; a DWU consists of one single wheel, which is kinematically equivalent to a single steerable wheel.

The architecture of WMRs with conventional, both centred and offset, orientable wheels is systematized and analyzed in [1, 2] based on the concepts of "degree of mobility" and "degree of steerability" introduced therein. In [3], the kinematic analysis of mobile robots equipped with centred orientable wheels was extended to singular configurations, which occur when two wheel planes are perpendicular to their line of centres, or when the instantaneous centre of rotation of the robot coincides with the centre of a wheel. Legrand and Slater [4] patented a robot with four powered offset wheels. Mori et al. [5] described an omnidirectional vehicle equipped with four conventional wheels with an offset, where each wheel unit is equipped with a motor for driving and steering. These WMRs have a mobility of three; they are hence said to be *of full mobility*. This means that these robots can move arbitrarily on a flat floor, and are hence called omnidirectional. A clear definition of the term *omnidirectional* can be found in [6], namely, "the ability of a system to move instantaneously in any direction from any configuration."

Kim et al. [7] derived the kinematic models of WMRs with different types of conventional wheels, with consideration of skidding and sliding to match the dimensions of the input and output vectors of each serial subchain of the robots. Park et al. [8] reported an attempt to find an optimum design of omni-directional mobile robots equipped with offset conventional wheels. Although the authors stated that "For the mobile robot to have omni-directional characteristics on the plane, only wheels with three degrees of freedom must be employed in mobile robots", this proves to be not so. In fact, if the platform of a WMR is equipped with three or more conventional orientable wheels (with or without an offset), which, being fixed to the platform by a revolute, have two degrees of freedom, the WMR at hand has a mobility of three and is omnidirectional.

It is stated in [6] that vehicles with steerable wheels must have at least two active wheels, each of which has both driving and steering actuators. Nevertheless, in this case the system has more actuators than its mobility, and hence, the actuators should be controlled in a coordinated manner.

In [9] the "velocity kinematic modeling" of WMRs is formulated by applying a matrix coordinate transformation to every pairs of WMRs including wheel pairs. An example is given for a three-wheeled platform with conventional wheels and, clearly, with an offset. In [10] the kinematics of three-wheeled mobile robots (3WMR) with conventional wheels is discussed. The relationship between the platform of the robot motion and the driving and steering rates is formulated based on the conditions of rolling on a horizontal plane without slipping. The degree of freedom of this robot is analyzed using the Freudenstein functional matrix [11].

The paper is organized into four main sections. In Section 2 we derive the geometric constraints imposed onto the system and defined the generalized coordinates describing uniquely its configuration. In Section 3 we obtain the kinematic constraints and compatibility conditions as well as the mobility of WMR with s wheels. Sections 4 and 5 are devoted to the direct and inverse kinematic relations and actuation schemes for four possible combinations of the numbers of centred and offset wheels.

2 Geometric Constraints and Generalized Coordinates

Mechanical systems are subjects to constraints. The presence of these constraints limits the motion of the system, thereby rendering the system suitable for a specific class of tasks. *Geometric* constraints in the domain of the generalized coordinates $\mathbf{q}$ are relations that can be represented in the form $\mathbf{f}(\mathbf{q}, t) = \mathbf{0}$, where t is time, while *kinematic* constraints are functions not only of the n-dimensional vector of generalized coordinates $\mathbf{q}$ and time t, but also of the generalized velocities $\dot{\mathbf{q}}$. Kinematic constraints being linear in the generalized velocities, they can be written in the form

$$\mathbf{A}(\mathbf{q}, t)\dot{\mathbf{q}} = \mathbf{b}(\mathbf{q}, t)$$

where $\mathbf{A}$ is a $p \times n$ matrix, while $\mathbf{b}$ is a p-dimensional vector. Apparently, $p < n$ in the above equation.

The wheeled mobile robots of interest to this paper are mechanical systems composed of a platform and $s \geq 3$ orientable single wheels or dual-wheel units. DWUs are known to be kinematically equivalent to single steerable wheels. We regard these mechanical systems as composed of rigid bodies connected by ideal joints and rolling on a plane, the wheel-ground contact being the contact point. We assume, moreover, that the friction at such contacts is large enough to prevent slipping and skidding [1]. This gives rise to kinematic constraints which are, in fact, linear in the generalized velocities and can be proven to be nonholonomic.

To provide the detailed kinematic analysis of this system, let us define an inertial frame $\mathcal{F}_0$ with the orthonormal triad of constant vectors $\{\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0\}$, $\mathbf{k}_0$ pointing in the upward direction normal to the plane of rolling; likewise, we define a moving frame $\mathcal{F}$ with the orthonormal diad $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, fixed to the platform, with origin at the mass centre of the platform, as depicted in Fig. 1.

Let us denote by $\mathbf{c}$ a two-dimensional projection onto the platform of the position vector of the mass centre C of the platform with respect to the frame $\mathcal{F}_0$. The pose of the platform, which undergoes planar motion, can be described by a three-dimensional array $\mathbf{q}_p \equiv [\ \mathbf{c}^T \quad \psi\ ]^T$, where ψ is the angle of the platform orientation with respect to the inertial frame $\mathcal{F}_0$. The pose of the ith wheel-unit can be described by a four-dimensional array $\mathbf{q}_i \equiv [\ \mathbf{o}_i^T \quad \varphi_i \quad \theta_i\ ]^T$, where $\mathbf{o}_i$ is the two-dimensional projection onto the platform of the position vector of the centre of the wheel, O_i, in $\mathcal{F}_0$, θ_i is the angle of rotation of the wheel about its axis, and $\varphi_i \equiv \phi_i + \psi$ where ϕ_i is the steering angle of the ith wheel, for $i = 1, \ldots, s$.

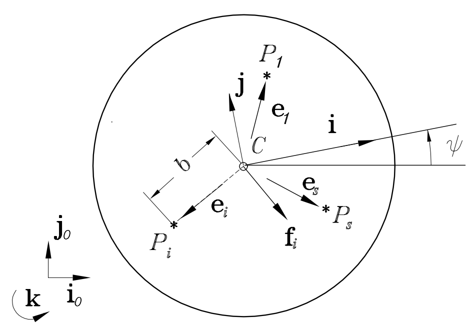

Figure 1: Unit vectors

Moreover, if the wheel units are dual-wheels (with or without an offset), then O_i is the midpoint of the line linking the centres of the wheels, $\varphi_i = (r/l)(\theta_{i1} - \theta_{i2})$, and $\theta_i = (1/2)(\theta_{i1} + \theta_{i2})$, θ_{i1} and θ_{i2} being the angles of rotation of the wheels about their respective axes, while r is the radius of the wheels and l is the length of the axle connecting the wheel centres.

Let us introduce, moreover, mutually orthogonal unit vectors $\mathbf{e}_i$ and $\mathbf{f}_i$ of the ith frame fixed to the platform, whose origin lies at the mass centre C of the platform, with $\mathbf{e}_i$ oriented toward the point P_i of intersection of the ith wheel-unit steering axis with the platform, as shown in Fig. 1. Moreover, mutually orthogonal unit vectors $\boldsymbol{\xi}_i$ and $\boldsymbol{\eta}_i$ are defined with the origin at the ith wheel centre O_i fixed to the ith wheel, as shown in Fig. 2. As a rule, wheels are connected to the platform at P_i by revolute joints.

If $\mathbf{p}_i$ is the horizontal component of the position vector of the point P_i, while $b = \overline{CP_i}$, then we have, with $\mathbf{c}$ denoting the two-dimensional position vector of C in $\mathcal{F}_0$,

$$\mathbf{p}_i = \mathbf{c} + b\mathbf{e}_i \quad i = 1, \ldots s. \tag{1}$$

On the other hand, $\mathbf{p}_i$ can be expressed in terms of the position vector O_i as

$$\mathbf{p}_i = \mathbf{o}_i - l\boldsymbol{\xi}_i \quad i = 1, \ldots, s \tag{2}$$

where l is the offset. Substituting eq.(2) into eq.(1), we obtain

$$\mathbf{o}_i = \mathbf{c} + b\mathbf{e}_i + l\boldsymbol{\xi}_i, \quad i = 1, \ldots, s \ . \tag{3}$$

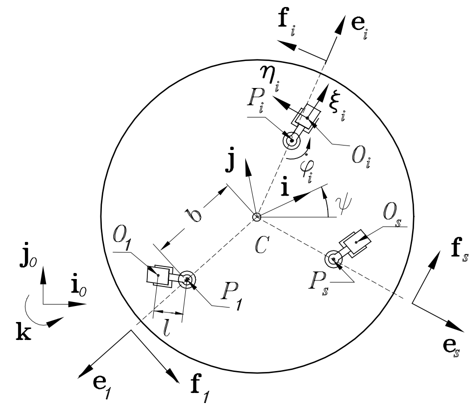

Figure 2: A WMR with conventional offset-wheels.

Hence, the mechanical system, which consists of the platform with s orientable wheel units, is subject to $2s$ geometric constraints (3); its configuration can be described by $3+4s-2s = 3+2s$ variables such as $[\ \mathbf{c}^T \quad \psi \quad \boldsymbol{\theta}^T \quad \boldsymbol{\varphi}^T\ ]^T$, where $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}$ are s-dimensional vectors, namely,

$$\boldsymbol{\theta} \equiv \left[\ \theta_1, \quad \ldots, \quad \theta_s\ \right]^T, \quad \boldsymbol{\varphi} \equiv \left[\ \varphi_1, \quad \ldots, \quad \varphi_s\ \right]^T.$$

3 Kinematic Constraints

Due to the nonslipping and nonskidding assumptions, each wheel is subject to the kinematic constraints

$$\dot{\mathbf{o}}_i - r\dot{\theta}_i\boldsymbol{\xi}_i = \mathbf{0}_2\ , \quad i = 1, \ldots, s \tag{4}$$

where r is the wheel radius and $\mathbf{0}_2$ is the two-dimensional zero vector. Moreover, upon differentiating both sides of eqs.(2) and (3) with respect to time, we obtain

$$\dot{\mathbf{p}}_i = \dot{\mathbf{o}}_i - l\dot{\varphi}_i\boldsymbol{\eta}_i\ , \tag{5a}$$

$$\dot{\mathbf{o}}_i = \dot{\mathbf{c}} + b\dot{\psi}\mathbf{f}_i + l\dot{\varphi}_i\boldsymbol{\eta}_i\ , \quad i = 1, \ldots, s\ . \tag{5b}$$

Substituting eq.(4) into eq.(5a), we have

$$\dot{\mathbf{p}}_i = r\dot{\theta}_i\boldsymbol{\xi}_i - l\dot{\varphi}_i\boldsymbol{\eta}_i\ .$$

Furthermore, it is well known that a planar motion of a rigid body can be viewed as a rotation around an instantant centre (IC) of rotation. Hence, the velocity of any point of the platform, including points at which the wheels are connected to the platform, namely, P_i, at each instant is orthogonal to the line linking this point to the IC. Therefore, the lines orthogonal to $\dot{\mathbf{p}}_i$, or to a vector $r\dot{\theta}_i\boldsymbol{\xi}_i - l\dot{\varphi}_i\boldsymbol{\eta}_i$ for that matter, and passing through P_i, intersect at the IC. In the case of centred wheel units ($l = 0$), the wheel axes themselves intersect at the IC. On the other hand, it is known that if lines $\{\mathcal{L}_i\}_1^s$ in a plane intersect at a common point, i.e., if the lines form a planar pencil, then the determinant of the *homogeneous coordinates* of these planar lines vanishes [12, 13], i.e.,

$$\det \begin{bmatrix} A_i & B_i & C_i \\ A_j & B_j & C_j \\ A_k & B_k & C_k \end{bmatrix} = 0 \ , \quad i, j, k \in \{1, \cdots, s\} \ ,$$

where A_ι, B_ι, and C_ι are the coefficients of the normalized equation of the lines in a plane, namely, $A_\iota x + B_\iota y + C_\iota = 0$, $\iota = 1, \cdots, s$. This means that if the IC and, hence, the orientation angles of any two of the robot wheel axes φ_i, with, e.g., $i = 1, 2$, are known, then the orientation angles of the remaining $s - 2$ wheel axes should satisfy the relations

$$\det \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_k & B_k & C_k \end{bmatrix} = 0 \ , \quad k = 3, \ldots, s \ . \tag{6}$$

The $(s - 2)$ eqs.(6) also represent constraints and are called here the *compatibility conditions*. If the wheels have no offset ($l = 0$), then eqs.(6) represent geometrical constraints and, hence, the dof of the WMR with orientable wheels is, in fact, $3 + 2s - (s - 2) = 5 + s$. Nevertheless, since eqs.(6) are usually highly nonlinear in φ_i, $i = 1, \ldots, s$, it is not advisable to express the orientation of $s - 2$ wheels in terms of the orientation of two independently oriented wheel-units using eqs.(6) to exclude the corresponding angles from the set of generalized coordinates. Instead, it turns out to be more convenient to choose the generalized coordinate vector as a $(3 + 2s)$-dimensional vector

$$\mathbf{q} = \left[\begin{array}{cccc} \psi & \mathbf{c}^T & \boldsymbol{\theta}^T & \boldsymbol{\varphi}^T \end{array} \right]^T \ .$$

Then, one can differentiate eqs.(6) sidewise with respect to time and take into account the resulting equations, which are linear in the joint velocities $\dot{\varphi}_i$, as kinematic constraints. Nevertheless, note that for a WMR with single wheels these constraints are holonomic.

Substituting eq.(5b) into eq.(4), we derive the kinematic constraints, expressed in terms of generalized coordinates and their time-derivatives, in the form:

$$\dot{\mathbf{c}} + b\dot{\psi}\mathbf{f}_i = r\dot{\theta}_i\boldsymbol{\xi}_i - l\dot{\varphi}_i\boldsymbol{\eta}_i \quad i = 1, \ldots s \ . \tag{7}$$

It can be shown that, in general, eqs.(7) are nonintegrable, and hence, these constraints are nonholonomic. From eqs.(7) we can derive $2s$ scalar kinematic constraint equations which are linear in the generalized velocities. However, $(s - 2)$ of these equations must satisfy the compatibility conditions (6). Hence, only $2s - (s - 2) = s + 2$ of them are independent. The mobility of the mechanical system at hand is, then, $m \equiv n - p = 5 + s - (s+2) = 3$. Therefore, WMRs with $s \geq 3$ orientable wheels or DWUs, both centred and offset, are omnidirectional.

It is noteworthy that there is a basic difference between centred and offset wheels as well as DWUs. Offset wheels, also known as casters, whose axes of orientation are unactuated would nevertheless be forced, by friction, to rotate around their vertical axes so as to accommodate the compatibility conditions (6). On the contrary, centred wheels cannot reorient themselves naturally to satisfy eq.(6) without actuation. Hence, all centred wheels must be steerable, and their orientations must be controlled in a coordinated fashion; therefore, if the robot is equipped with more than two centred orientable wheels or DWUs without an offset, it should be overactuated.

4 Direct and Inverse Kinematics

Suppose the layout of the points at which the wheels are fixed to the platform is symmetric with respect to the platform centre. Then, by adding sidewise eqs.(7), we obtain

$$\dot{\mathbf{c}} = \frac{1}{s}\sum_{i=1}^{s}(r\dot{\theta}_i\boldsymbol{\xi}_i - l\dot{\varphi}_i\boldsymbol{\eta}_i) \ . \tag{8}$$

Likewise, upon dot-multiplying eqs.(7) by $\mathbf{f}_i$ and adding all s equations sidewise, we obtain

$$\dot{\psi} = \frac{1}{bs}\sum_{i=1}^{s}(r\dot{\theta}_i\mathbf{f}_i^T\boldsymbol{\xi}_i - l\dot{\varphi}_i\mathbf{f}_i^T\boldsymbol{\eta}_i) \ . \tag{9}$$

Then, the twist $\mathbf{t}$ of the platform can be expressed as

$$\mathbf{t} \equiv \begin{bmatrix} \dot{\mathbf{c}} \\ \dot{\psi} \end{bmatrix} = \frac{1}{bs}\sum_{i=1}^{s}\begin{bmatrix} b(r\dot{\theta}_i\boldsymbol{\xi}_i - l\dot{\varphi}_i\boldsymbol{\eta}_i) \\ \mathbf{f}_i^T(r\dot{\theta}_i\boldsymbol{\xi}_i - l\dot{\varphi}_i\boldsymbol{\eta}_i) \end{bmatrix} \ .$$

On the other hand, upon premultiplying eq.(7) by $\boldsymbol{\xi}_i^T$ and by $\boldsymbol{\eta}_i^T$, we obtain

$$r\dot{\theta}_i = \boldsymbol{\xi}_i^T(\dot{\mathbf{c}} + b\dot{\psi}\mathbf{f}_i) \ , \tag{10a}$$
$$l\dot{\varphi}_i = -\boldsymbol{\eta}_i^T(\dot{\mathbf{c}} + b\dot{\psi}\mathbf{f}_i) \ , \tag{10b}$$

which can be cast in the form

$$\dot{\boldsymbol{\theta}} = \frac{1}{r}\begin{bmatrix} \boldsymbol{\Xi}^T & b\mathbf{g} \end{bmatrix}\begin{bmatrix} \dot{\mathbf{c}} \\ \dot{\psi} \end{bmatrix} \ ,$$
$$\dot{\boldsymbol{\varphi}} = -\frac{1}{l}\begin{bmatrix} \boldsymbol{\Gamma}^T & b\mathbf{h} \end{bmatrix}\begin{bmatrix} \dot{\mathbf{c}} \\ \dot{\psi} \end{bmatrix} \ ,$$

where

$$\boldsymbol{\Xi} \equiv \begin{bmatrix} \boldsymbol{\xi}_1 & \boldsymbol{\xi}_2 & \cdots & \boldsymbol{\xi}_s \end{bmatrix} \ , \qquad \boldsymbol{\Gamma} \equiv \begin{bmatrix} \boldsymbol{\eta}_1 & \boldsymbol{\eta}_2 & \cdots & \boldsymbol{\eta}_s \end{bmatrix} \ ,$$
$$\mathbf{g} \equiv \begin{bmatrix} \boldsymbol{\xi}_1^T\mathbf{f}_1 & \boldsymbol{\xi}_2^T\mathbf{f}_2 & \cdots & \boldsymbol{\xi}_s^T\mathbf{f}_s \end{bmatrix}^T \ , \quad \mathbf{h} \equiv \begin{bmatrix} \boldsymbol{\eta}_1^T\mathbf{f}_1 & \boldsymbol{\eta}_2^T\mathbf{f}_2 & \cdots & \boldsymbol{\eta}_s^T\mathbf{f}_s \end{bmatrix}^T \ .$$

5 Actuation Scheme

Substituting eqs.(8) and (9) into eqs.(10a & b), we derive

$$\frac{1}{s}\sum_{k=1}^{s}(r\dot{\theta}_k\boldsymbol{\xi}_k - l\dot{\varphi}_k\boldsymbol{\eta}_k)^T\boldsymbol{\xi}_i + \frac{1}{s}\sum_{k=1}^{s}\mathbf{f}_k^T(r\dot{\theta}_k\boldsymbol{\xi}_k - l\dot{\varphi}_k\boldsymbol{\eta}_k)\mathbf{f}_i^T\boldsymbol{\xi}_i - r\dot{\theta}_i = 0\ ,$$

$$\frac{1}{s}\sum_{k=1}^{s}(r\dot{\theta}_k\boldsymbol{\xi}_k - l\dot{\varphi}_k\boldsymbol{\eta}_k)^T\boldsymbol{\eta}_i + \frac{1}{s}\sum_{k=1}^{s}\mathbf{f}_k^T(r\dot{\theta}_k\boldsymbol{\xi}_k - l\dot{\varphi}_k\boldsymbol{\eta}_k)\mathbf{f}_i^T\boldsymbol{\eta}_i + l\dot{\varphi}_i = 0\ .$$

The above equations can be rewritten in the form

$$\boldsymbol{\xi}_i^T\sum_{k=1}^{s}\mathbf{F}_{ik}(r\dot{\theta}_k\boldsymbol{\xi}_k - l\dot{\varphi}_k\boldsymbol{\eta}_k) - rs\dot{\theta}_i = 0\ , \qquad (11a)$$

$$\boldsymbol{\eta}_i^T\sum_{k=1}^{s}\mathbf{F}_{ik}(r\dot{\theta}_k\boldsymbol{\xi}_k - l\dot{\varphi}_k\boldsymbol{\eta}_k) + ls\dot{\varphi}_i = 0\ , \qquad (11b)$$

where we introduced the notation $\mathbf{F}_{ik} \equiv \mathbf{f}_i\mathbf{f}_k^T + \mathbf{1}_2$. Equations (11a & b) can be cast in the form

$$\begin{bmatrix} r\boldsymbol{\xi}^T\mathbf{F}\boldsymbol{\xi} - rs\mathbf{I}_{s\times s} & -l\boldsymbol{\xi}^T\mathbf{F}\boldsymbol{\eta} \\ r\boldsymbol{\eta}^T\mathbf{F}\boldsymbol{\xi}^T & -l\boldsymbol{\eta}^T\mathbf{F}\boldsymbol{\eta} + ls\mathbf{I}_{s\times s} \end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{\theta}} \\ \dot{\boldsymbol{\varphi}}\end{bmatrix} = \mathbf{0}\ ,$$

where

$$\boldsymbol{\xi} \equiv \begin{bmatrix}\boldsymbol{\xi}_1 \\ \vdots \\ \boldsymbol{\xi}_s\end{bmatrix}\ ,\quad \boldsymbol{\eta} \equiv \begin{bmatrix}\boldsymbol{\eta}_1 \\ \vdots \\ \boldsymbol{\eta}_s\end{bmatrix}\ ,\quad \mathbf{F} \equiv \begin{bmatrix}\mathbf{F}_{11} & \mathbf{F}_{12} & \cdots & \mathbf{F}_{1s} \\ \mathbf{F}_{21} & \mathbf{F}_{22} & \cdots & \mathbf{F}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{F}_{s1} & \mathbf{F}_{s2} & \cdots & \mathbf{F}_{ss}\end{bmatrix}\ .$$

Moreover, by multiplying eqs.(7) by $\mathbf{e}_i^T$ and adding them sidewise, we derive, under the symmetry assumption mentioned above,

$$\sum_{i=1}^{s}\mathbf{e}_i^T(r\dot{\theta}_i\boldsymbol{\xi}_i - l\dot{\varphi}_i\boldsymbol{\eta}_i) = 0$$

which can be simplified to

$$\sum_{i=1}^{s}[r\dot{\theta}_i\cos(\varphi_i - \psi) - l\dot{\varphi}_i\sin(\varphi_i - \psi)] = 0\ .$$

Suppose that the WMR at hand has σ centred and $s-\sigma$ offset orientable wheels. Let us now consider four particular cases: (i) $\sigma = 0$, i.e., a WMR with only offset wheels; (ii) $\sigma = 1$; hence, a WMR with one centred wheel and $(s-1)$ offset wheels; (iii) $\sigma \geq 2$, i.e., a WMR with two or more centred wheels; and (iv) $\sigma = s$, i.e., a WMR with only centred steerable wheels.

(i) If $\sigma = 0$, then the independent kinematic constraints are

$$\dot{\mathbf{c}}^T\boldsymbol{\xi}_i + b\dot{\psi}\mathbf{f}_i^T\boldsymbol{\xi}_i - r\dot{\theta}_i = 0\ , \quad i = 1, 2, \cdots, s,$$

$$\dot{\mathbf{c}}^T\boldsymbol{\eta}_k + b\dot{\psi}\mathbf{f}_k^T\boldsymbol{\eta}_k + l\dot{\varphi}_k = 0\ , \quad k = 1, 2\ .$$

These equations can be cast in the form

$$\mathbf{A}_1\dot{\mathbf{q}} = \mathbf{0} \tag{12}$$

where $\dot{\mathbf{q}}$ is the $(5+s)$-dimensional vector defined as

$$\dot{\mathbf{q}} \equiv \begin{bmatrix} \dot{\mathbf{c}}^T & \dot{\psi} & \dot{\theta}_1 & \cdots & \dot{\theta}_s & \dot{\varphi}_1 & \dot{\varphi}_2 \end{bmatrix}^T$$

and $\mathbf{A}_1$ is the $(s+2)\times(s+5)$ kinematic constraint matrix:

$$\mathbf{A}_1 \equiv \begin{bmatrix} b\mathbf{f}_1^T\boldsymbol{\xi}_i & \boldsymbol{\xi}_i^T & -r & 0 & \cdots & 0 & 0 & 0 \\ b\mathbf{f}_2^T\boldsymbol{\xi}_i & \boldsymbol{\xi}_i^T & 0 & -r & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ b\mathbf{f}_s^T\boldsymbol{\xi}_s & \boldsymbol{\xi}_s^T & 0 & 0 & \cdots & -r & 0 & 0 \\ b\mathbf{f}_1^T\boldsymbol{\eta}_1 & \boldsymbol{\eta}_1^T & 0 & 0 & \cdots & 0 & l & 0 \\ b\mathbf{f}_2^T\boldsymbol{\eta}_2 & \boldsymbol{\eta}_2^T & 0 & 0 & \cdots & 0 & 0 & l \end{bmatrix}$$

If $\varphi_1 \neq \varphi_2$, this matrix is of full rank, which means that one $\dot{\theta}_i$ and the two components of $\dot{\boldsymbol{\varphi}}$, namely, $\dot{\varphi}_1$ and $\dot{\varphi}_2$, should be actuated independently.

(*ii*) If $\sigma = 1$, then the system of independent kinematic constraints is

$$\dot{\mathbf{c}}^T\boldsymbol{\xi}_i + b\dot{\psi}\mathbf{f}_i^T\boldsymbol{\xi}_i - r\dot{\theta}_i = 0\ , \quad i = 1,\cdots,s;$$
$$\dot{\mathbf{c}}^T\boldsymbol{\eta}_1 + b\dot{\psi}\mathbf{f}_1^T\boldsymbol{\eta}_1 = 0\ ,$$
$$\dot{\mathbf{c}}^T\boldsymbol{\eta}_2 + b\dot{\psi}\mathbf{f}_2^T\boldsymbol{\eta}_2 + l\dot{\varphi}_2 = 0\ ,$$

which can be cast in the form

$$\mathbf{A}_2\dot{\mathbf{q}} = \mathbf{0}$$

where $\mathbf{A}_2$ is the corresponding $(s+2)\times(5+s)$ kinematic constraint matrix:

$$\mathbf{A}_2 \equiv \begin{bmatrix} b\mathbf{f}_1^T\boldsymbol{\xi}_1 & \boldsymbol{\xi}_1^T & -r & 0 & \cdots & 0 & 0 \\ b\mathbf{f}_2^T\boldsymbol{\xi}_2 & \boldsymbol{\xi}_2^T & 0 & -r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b\mathbf{f}_s^T\boldsymbol{\xi}_s & \boldsymbol{\xi}_s^T & 0 & 0 & \cdots & -r & 0 \\ b\mathbf{f}_1^T\boldsymbol{\eta}_1 & \boldsymbol{\eta}_1^T & 0 & 0 & \cdots & 0 & 0 \\ b\mathbf{f}_2^T\boldsymbol{\eta}_2 & \boldsymbol{\eta}_2^T & 0 & 0 & \cdots & 0 & l \end{bmatrix}.$$

If $\varphi_1 \neq \varphi_2$, this matrix is of full rank, which means that one $\dot{\theta}_i$ and the two components of $\dot{\boldsymbol{\varphi}}$, i.e., $\dot{\varphi}_1$ and $\dot{\varphi}_2$, can be actuated independently. It makes sense hence to actuate the orientation of the only centred wheel and that of one of the offset wheels.

(*iii*) If $\sigma \geq 2$, then all centred wheels must be steerable, but only two of those wheel orientations can be actuated independently, while φ_κ and $\dot{\varphi}_\kappa$ for $\kappa = \sigma+1,\cdots,s$, must be steered to satisfy the compatibility conditions (6). Then, the set of independent kinematic constraints is

$$\dot{\mathbf{c}}^T\boldsymbol{\xi}_i + b\dot{\psi}\mathbf{f}_i^T\boldsymbol{\xi}_i - r\dot{\theta}_i = 0 \ , \quad i = 1,\ldots,s;$$
$$\dot{\mathbf{c}}^T\boldsymbol{\eta}_k + b\dot{\psi}\mathbf{f}_k^T\boldsymbol{\eta}_k = 0 \ , \quad k = 1,2$$

which can be cast in the form

$$\mathbf{A}_3\dot{\mathbf{q}} = \mathbf{0}$$

where $\dot{\mathbf{q}}$ is the $(3+s)$-dimensional vector

$$\dot{\mathbf{q}} \equiv [\ \dot{\mathbf{c}}^T \quad \dot{\psi} \quad \dot{\theta}_1 \quad \cdots \quad \dot{\theta}_s\ ]^T$$

and $\mathbf{A}_3$ is the corresponding $(s+2)\times(s+3)$ kinematic constraint matrix:

$$\mathbf{A}_3 \equiv \begin{bmatrix} b\mathbf{f}_1^T\boldsymbol{\xi}_1 & \boldsymbol{\xi}_1^T & -r & 0 & \cdots & 0 \\ b\mathbf{f}_2^T\boldsymbol{\xi}_2 & \boldsymbol{\xi}_2^T & 0 & -r & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b\mathbf{f}_s^T\boldsymbol{\xi}_s & \boldsymbol{\xi}_s^T & 0 & 0 & \cdots & -r \\ b\mathbf{f}_1^T\boldsymbol{\eta}_1 & \boldsymbol{\eta}_1^T & 0 & 0 & \cdots & 0 \\ b\mathbf{f}_2^T\boldsymbol{\eta}_2 & \boldsymbol{\eta}_2^T & 0 & 0 & \cdots & 0 \end{bmatrix} .$$

If $\varphi_1 \neq \varphi_2$, this matrix is of full rank, which means that only one $\dot{\theta}_i$ can be actuated independently.

(*iv*) If $\sigma = s$, all wheels are centered. Then, the kinematic constraint eqs.(7) change to

$$\dot{\mathbf{c}} + b\dot{\psi}\mathbf{f}_i - r\dot{\theta}_i\boldsymbol{\xi}_i = \mathbf{0}_2 \ , \quad i = 1,\cdots,s \ . \tag{13}$$

Since the steering rates $\dot{\varphi}_i$ do not appear in eq.(13), the independent generalized velocity vector can be defined as $\dot{\mathbf{q}} \equiv \begin{bmatrix} \dot{\mathbf{c}}^T & \dot{\psi} & \dot{\theta}_1 & \cdots & \dot{\theta}_s \end{bmatrix}^T$ and the $2s\times(s+3)$ constraint matrix $\mathbf{A}_4$ is

$$\mathbf{A}_4 \equiv \begin{bmatrix} b\mathbf{f}_1 & \mathbf{1}_2 & -r\boldsymbol{\xi}_1 & \mathbf{0}_2 & \cdots & \mathbf{0}_2 \\ b\mathbf{f}_2 & \mathbf{1}_2 & \mathbf{0}_2 & -r\boldsymbol{\xi}_2 & \cdots & \mathbf{0}_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b\mathbf{f}_s & \mathbf{1}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \cdots & -r\boldsymbol{\xi}_s \end{bmatrix} .$$

If, for example, the platform has three centred orientable wheels ($\delta = s = 3$) and matrix $\mathbf{A}_4$ is of full rank, then the system of eqs.(13) admits only the trivial solution. This means, as was shown in [14], that if the wheels do not have an offset and all $\dot{\theta}_i$ are independent, the system being unable to move, unless this matrix is rank-deficient. For example, if the layout of the wheels is symmetric, then the rank of matrix $\mathbf{A}_4$ is, at most, five, and its determinant vanishes, which leads to relation (5.33) in [14]:

$$2[\sin(2\varphi_1) + \sin(2\varphi_2) + \sin(2\varphi_3)] + \sin 2(-\varphi_1 + \varphi_2 + \varphi_3)$$
$$+ \sin 2(\varphi_1 - \varphi_2 + \varphi_3) + \sin 2(\varphi_1 + \varphi_2 - \varphi_3) = 0 \ .$$

This gives one scalar equation for the three variables φ_1, φ_2, and φ_3; therefore, two of them, i.e., φ_1 and φ_2, can be actuated independently. Nevertheless, it is obvious that the reaction forces (friction, in fact) at the contact point of the wheels with the ground cannot rotate the wheel plane. Hence, the third wheel orientation angle φ_3 should also be actuated, although not independently, but rather in a coordinated fashion. Moreover, since the mobility of the system is $m = 3$ and two orientation angles of the wheels, i.e., φ_1, φ_2, are independent, we are left with only one independent generalized velocity, which can be chosen as one of the three angular velocities $\dot{\theta}_i$, $i = 1, 2, 3$.

6 Conclusions

In general, based on the number and type of wheel units, WMRs have a number of independent generalized coordinates different from their degree of freedom (dof). Moreover, even if the dof of a WMR coincides with its number of independent generalized coordinates, the WMR may have a different number of independent generalized velocities, which define the *mobility* of the system. However, for any WMR equipped with conventional or dual wheels, whose platform undergoes planar motion, the number of independent generalized velocities, or the mobility of the system, is three, notwithstanding its dof, which usually is bigger that three.

The direct and inverse kinematic relations for these robots were obtained as well as the pertinent *compatibility conditions*.

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